CHAPTER 1

FUNDAMENTALS

1.1. INTRODUCTION

Man's desire for knowledge of fluid phenomena began with his problems of water supply, irrigation, navigation, and waterpower.

Matter exists in two states; the solid and the fluid, the fluid state being commonly divided into the liquid and gaseous states. Solids differ from liquids and liquids from gases in the spacing and latitude of motion of their molecules, these variables being large in a gas, smaller in a liquid, and extremely small in a solid. Thus it follows that intermolecular cohesive forces are large in a solid, smaller in a liquid, and extremely small in a gas.

1.2. DIMENSIONS AND UNITS

Dimension = A dimension is the measure by which a physical variable is expressed quantitatively.

Unit = A unit is a particular way of attaching a number to the quantitative dimension.

Thus length is a dimension associated with such variables as distance, displacement, width, deflection, and height, while centimeters or meters are both numerical units for expressing length.

In fluid mechanics, there are only four primary dimensions from which all the dimensions can be derived: mass, length, time, and force. The brackets around a symbol like [M] mean "the dimension" of mass. All other variables in fluid mechanics can be expressed in terms of [M], [L], [T], and [F]. For example, acceleration has the dimensions [LT^{-2}]. Force [F] is directly related to mass, length, and time by Newton's second law,

$$\begin{aligned} F &= ma \\ Force &= Mass \times Acceleration \end{aligned} \qquad (1.1)$$

From this we see that, dimensionally, [F] = [MLT^{-2}].

1 kg-force = 9.81 Newton of force = 9.81 N

Primary Dimensions in SI and MKS Systems

Primary Dimension	MKS Units	SI Units
Force [F]	Kilogram (kg)	Newton ($N=kg.m/s^2$)
Mass [M]	$M=G/g = (kgsec^2/m)$	Kilogram
Length [L]	Meter (m)	Meter (m)
Time [T]	Second (sec)	Second (sec)

Secondary Dimensions in Fluid Mechanics

Secondary Dimension	MKS Units	SI Units
Area $[L^2]$	m^2	m^2
Volume $[L^3]$	m^3	m^3
Velocity $[LT^{-1}]$	m/sec	m/sec
Acceleration $[LT^{-2}]$	m/sec^2	m/sec^2
Pressure or stress $[FL^{-2}] = [ML^{-1}T^{-2}]$	kg/m^2	Pa= N/m^2(Pascal)
Angular Velocity $[T^{-1}]$	sec^{-1}	sec^{-1}
Energy, work $[FL] = [ML^2T^{-2}]$	kg.m	J = Nm (Joule)
Power $[FLT^{-1}] = [ML^2T^{-3}]$	kg.m/sec	W = J/sec (Watt)
Specific mass (ρ) $[ML^{-3}] = [FT^2L^{-4}]$	$kg.sec^2/m^4$	kg/m^3
Specific weight (γ) $[FL^{-3}] = [ML^{-2}T^{-2}]$	Kg/m^3	N/m^3

Specific mass = ρ = The mass, the amount of matter, contained in a volume. This will be expressed in mass-length-time dimensions, and will have the dimensions of mass [M] per unit volume $[L^3]$. Thus,

$$SpecificMass = \frac{Mass}{Volume}$$

$$[\rho]=\left[\frac{M}{L^3}\right]=\left[\frac{FT^2}{L^4}\right],\left(kg\sec^2/m^4\right)$$

Specific weight= γ = will be expressed in force-length-time dimensions and will have dimensions of force [F] per unit volume [L^3].

$$Specificweight=\frac{Weight}{Volume}$$

$$[\gamma]=\left[\frac{F}{L^3}\right]=\left[\frac{M}{L^2T^2}\right],\left(kg/m^3\right)$$

Because the weight (a force), W, related to its mass, M, by Newton's second law of motion in the form

$$W=Mg$$

In which g is the acceleration due to the local force of gravity, specific weight and specific mass will be related by a similar equation,

$$\gamma=\rho g \qquad (1.2)$$

EXAMPLE 1.1: Specific weight of the water at 4°C temperature is γ = 1000 kg/m^3. What is its the specific mass?

SOLUTION:

$$\gamma=\rho g=1000\left(kg/m^3\right)$$

$$\rho=\frac{1000}{9.81}=101.94\left(kg\sec^2/m^4\right)$$

EXAMPLE 1.2: A body weighs 1000 kg when exposed to a standard earth gravity g = 9.81 m/sec^2. a) What is its mass? b) What will be the weight of the body be in Newton if it is exposed to the Moon's standard acceleration g_{moon} = 1.62 m/sec^2? c) How fast will the body accelerate if a net force of 100 kg is applied to it on the Moon or on the Earth?

SOLUTION:

a) Since,

$$W=mg=1000(kg)$$

$$M=\frac{W}{g}=\frac{1000}{9.81}=101.94\left(kg\sec^2/m^4\right)$$

b) The mass of the body remains 101.94 kgsec^2/m regardless of its location. Then,

$$W = mg = 101.94 \times 1.62 = 165.14(kg)$$

In Newtons,

$$165.14 \times 9.81 = 1620(Newton)$$

c) If we apply Newton's second law of motion,

$$F = ma = 100(kg)$$

$$a = \frac{100}{101.94} = 0.98\left(m/\sec^2\right)$$

This acceleration would be the same on the moon or earth or anywhere.

All theoretical equations in mechanics (and in other physical sciences) are *dimensionally homogeneous*, i.e.; each additive term in the equation has the same dimensions.

EXAMPLE 1.3: A useful theoretical equation for computing the relation between the pressure, velocity, and altitude in a steady flow of a nearly inviscid, nearly incompressible fluid is the Bernoulli relation, named after Daniel Bernoulli.

$$p_0 = p + \frac{1}{2}\rho V^2 + \rho g z$$

Where
p_0 = Stagnation pressure
p = Pressure in moving fluid
V = Velocity
ρ = Specific mass
z = Altitude
g = Gravitational acceleration

a) Show that the above equation satisfies the principle of dimensional homogeneity, which states that all additive terms in a physical equation must have the same dimensions. b) Show that consistent units result in MKS units.

SOLUTION:

a) We can express Bernoulli equation dimensionally using brackets by entering the dimensions of each term.

$$[p_0] = [p] + \frac{1}{2}\left[\rho V^2\right] + [\rho g z]$$

The factor ½ is a pure (dimensionless) number, and the exponent 2 is also dimensionless.

$$\left[FL^{-2}\right]=\left[FL^{-2}\right]+\left[FT^{2}L^{-4}\right]\left[L^{2}T^{-2}\right]+\left[FT^{2}L^{-4}\right]\left[LT^{-2}\right]\left[L\right]$$

$$\left[FL^{-2}\right]=\left[FL^{-2}\right]$$

For all terms.

b) If we enter MKS units for each quantity:

$$\left(kg/m^{2}\right)=\left(kg/m^{2}\right)+\left(kg\sec^{2}/m^{4}\right)\left(m^{2}/\sec^{2}\right)+\left(kg\sec^{2}/m^{4}\right)\left(m/\sec^{2}\right)\left(m\right)$$

$$=\left(kg/m^{2}\right)$$

Thus all terms in Bernoulli's equation have units in kilograms per square meter when MKS units are used.

Many empirical formulas in the engineering literature, arising primarily from correlation of data, are dimensional inconsistent. Dimensionally inconsistent equations, though they abound in engineering practice, are misleading and vague and even dangerous, in the sense that they are often misused outside their range of applicability.

EXAMPLE 1.4: In 1890 Robert Manning proposed the following empirical formula for the average velocity V in uniform flow due to gravity down an open channel.

$$V=\frac{1}{n}R^{2/3}S^{1/2}$$

Where

R = Hydraulics radius of channel
S= Channel slope (tangent of angle that bottom makes with horizontal)
n = Manning's roughness factor

And n is a constant for a given surface condition for the walls and bottom of the channel. Is Manning's formula dimensionally consistent?

SOLUTION: Introduce dimensions for each term. The slope S, being a tangent or ratio, is dimensionless, denoted by unity or $[F^0L^0T^0]$. The above equation in dimensional form

$$\left[\frac{L}{T}\right]=\left[\frac{1}{n}\right]\left[L^{2/3}\right]\left[F^{0}L^{0}T^{0}\right]$$

This formula cannot be consistent unless $[L/T] = [L^{1/3}/T]$. In fact, Manning's formula is inconsistent both dimensionally and physically and does not properly account for channel-roughness effects except in a narrow range of parameters, for water only.

Engineering results often are too small or too large for the common units, with too many zeros one way or the other. For example, to write F = 114000000 ton is long and awkward. Using the prefix "M" to mean 10^6, we convert this to a concise F = 114 Mton (megatons). Similarly, t = 0.000003 sec is a proofreader's nightmare compared to the equivalent t = 3 μsec (microseconds)

TABLE 1.1
CONVENIENT PREFIXES FOR ENGINEERING UNITS

Multiplicative Factor	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

1.3 MOLECULAR STRUCTURE OF MATERIALS

Solids, liquids and gases are all composed of molecules in continuous motion. However, the arrangement of these molecules, and the spaces between them, differ, giving rise to the characteristics properties of the three states of matter. In solids, the molecules are densely and regularly packed and movement is slight, each molecule being strained by its neighbors. In liquids, the structure is loser; individual molecules have greater freedom of movement and, although restrained to some degree by the surrounding molecules, can break away from the restraint, causing a change of structure. In gases, there is no formal structure, the spaces between molecules are large and the molecules can move freely.

In this book, fluids will be assumed to be continuous substances, and, when the behavior of a small element or particle of fluid is studied, it will be assumed that it contains so many molecules that it can be treated as part of this continuum. Quantities such as velocity and pressure can be considered to be constant at any point, and changes due to molecular motion may be ignored. Variations in such quantities can also be assumed to take place smoothly, from point to point.

1.4. COMPRESSIBILITY: BEHAVIOR OF FLUIDS AGAINST PRESSURE

For most purposes a liquid may be considered as incompressible. The compressibility of a liquid is expressed by its *bulk modulus of elasticity*. The mechanics of compression of a fluid may be demonstrated by imagining the cylinder and piston of Fig.1.1 to be perfectly rigid (inelastic) and to contain a volume of fluid V. Application of a force, F, to piston will increase the pressure, p, in the fluid and cause the volume decrease –dV. The bulk modulus of elasticity, E, for the volume V of a liquid

$$E = -\frac{dp}{dV/V} \tag{1.3}$$

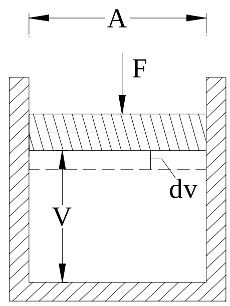


Fig. 1.1

Since dV/V is dimensionless, E is expressed in the units of pressure, p. For water at ordinary temperatures and pressures, $E = 2\times10^4$ kg/cm^2.

For liquids, the changes in pressure occurring in many fluid mechanics problems are not sufficiently great to cause appreciable changes in specific mass. It is, therefore, usual to ignore such changes and to treat liquids as incompressible.

ρ = Constant

1.5. VISCOSITY: BEHAVIOR OF FLUIDS AGAINST SHEAR STRESS

When real fluid motions are observed carefully, two basic types of motion are seen. The first is a smooth motion in which fluid elements or particles appear to slide over each other in layers or laminae; this motion is called *laminar* flow. The second distinct motion that occurs is characterized by a random or chaotic motion of individual particles; this motion is called *turbulent* flow.

Now consider the laminar motion of a real fluid along a solid boundary as in Fig. 1.2. Observations show that, while the fluid has a finite velocity, u, at any finite distance from the boundary, there is no velocity at the boundary. Thus, the velocity increases with increasing distance from the boundary. These facts are summarized on the *velocity profile*, which indicates relative motion between adjacent layers. Two such layers are showing having thickness dy, the lower layer moving with velocity u, the upper with velocity u+du. Two particles 1 and 2, starting on the same vertical line, move different distances d_1 = udt and d_2 = (u+du) dt in an infinitesimal time dt.

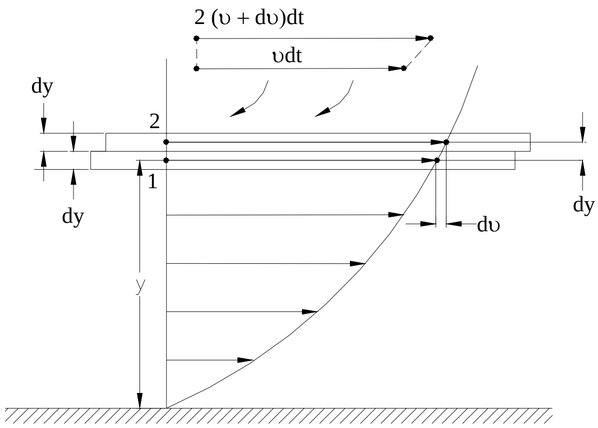


Fig. 1.2

It is evident that a frictional or shearing force must exist between the fluid layers; it may be expressed as a *shearing* or *frictional stress* per unit of contact area. This stress, designated by τ, has been found for laminar (nonturbulent) motion to be proportional to the velocity gradient, du/dy, with a constant of proportionality, μ, defined as *coefficient of viscosity* or *dynamic viscosity*. Thus,

$$\tau = \mu \frac{du}{dy} \tag{1.4}$$

All real fluids possess viscosity and therefore exhibit certain frictional phenomena when motion occurs. Viscosity results fundamentally from cohesion and molecular momentum exchange between fluid layers and, as flow occurs, these effects appear as tangential or shearing stresses between the moving layers. This equation is called as *Newton's law of viscosity*.

Because Equ. (1.4) is basic to all problems of fluid resistance, its implications and restrictions are to be emphasized:

1) The nonappearance of pressure in the equation shows that both τ and μ are independent of pressure, and that therefore fluid friction is different from that between moving solids, where plays a large part,
2) Any shear stress τ, however small, will cause flow because applied tangential forces must produce a velocity gradient, that is, relative motion between adjacent fluid layers,
3) Where du/dy = 0, τ = 0, regardless of the magnitude of μ, the shearing stress in viscous fluids at rest will be zero,
4) The velocity profile cannot be tangent to a solid boundary because this would require an infinite velocity gradient and infinite shearing stress between fluids and solids,
5) The equation is limited to nonturbulent (laminar) fluid motion, in which viscous action is strong.
6) The velocity at a solid boundary is zero, that is, there is no slip between fluid and solid for all fluids that can be treated as a continuum.

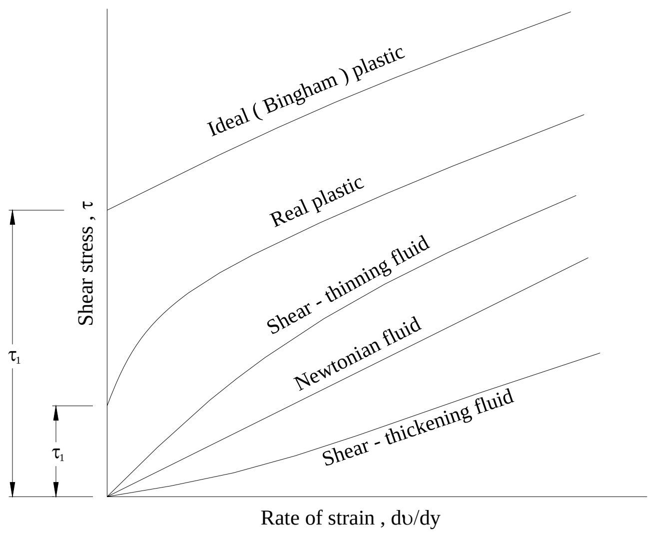


Fig. 1.3

Equ. (1.4) may be usefully visualized on the plot of Fig.1.3 on which μ is the slope of a straight line passing through the origin, here du will be considered as displacement per unit time and the velocity gradient du/dy as time of strain. Fluids that follow Newton's viscosity law are commonly known as *Newtonian* fluids. It is these fluids with which this book is concerned. Other fluids are classed as *non-Newtonian* fluids. The science of Rheology, which broadly is the study of the deformation and flow of matter, is concerned with plastics, blood, suspensions, paints, and foods, which flow but whose resistance is not characterized by Equ. (1.4).

The dimensions of the (dynamic) viscosity μ may be determined from dimensional homogeneity as follows:

$$[\mu] = \frac{[\tau]}{[du/dy]} = \frac{[FL^{-2}]}{[LT^{-1}L^{-1}]} = [FL^{-2}T] \quad , \ (kg.\sec/m^2)$$

In SI units, (Pa×sec). These combination times 10^{-1} is given the special name *poises*.

Viscosity varies widely with temperature. The shear stress and thus the viscosity of gases will increase with temperature. Liquid viscosities decrease as temperature rises.

Owing to the appearance of the ratio μ/ρ in many of the equations of the fluid flow, this term has been defined by,

$$\vartheta = \frac{\mu}{\rho} \tag{1.5}$$

in which υ is called the *kinematic viscosity*. Dimensional considerations of Equ. (1.5) shows the units of υ to be square meters per second, a combination of kinematic terms, which explains the name kinematic viscosity. The dimensional combination times 10^{-4} is known as *stokes*.

1.6.VAPOR PRESSURE AND CAVITATION

Vapor pressure is the pressure at which a liquid boils and is in equilibrium with its own vapor. For example, the vapor pressure of water at 10^0C is 0.125 t/m^2, and at 40^0C is 0.75 t/m^2. If the liquid pressure is greater than the vapor pressure, the only exchange between liquid and vapor is evaporation at the interface. If, however, the liquid pressure falls below the vapor pressure, vapor bubbles begin to appear in the liquid. When the liquid pressure is dropped below the vapor pressure due to the flow phenomenon, we call the process *cavitation*. Cavitation can cause serious problems, since the flow of liquid can sweep this cloud of bubbles on into an area of higher pressure where the bubbles will collapse suddenly. If this should occur in contact with a solid surface, very serious damage can result due to the very large force with which the liquid hits the surface. Cavitation can affect the performance of hydraulic machinery such as pumps, turbines and propellers, and the impact of collapsing bubbles can cause local erosion of metal surfaces.

CHAPTER 2

FLUID STATICS

Fluid statics is the study of fluid problems in which there is no relative motion between fluid elements. With no relative motion between individual elements (and thus no velocity gradients), no shear can exist, whatever the viscosity of the fluid is. Accordingly, viscosity has no effect in static problems and exact analytical solutions to such problems are relatively easy to obtain. Hence, all free bodies in fluid statics have only normal pressure forces acting on them.

2.1. PRESSURE AT A POINT

The average pressure is calculated by dividing the normal force pushing against a plate area by the area. The pressure at a point is the limit of the ratio of normal force to area, as the area approaches zero size at the point.

Fig. 2.1 shows a small wedge of fluid at rest of size Δx by Δz by Δs and depth b into the paper. Since there can be no shear forces, the only forces are the normal surface forces and gravity. Summation of forces must equal zero (no acceleration) in both the x and z directions.

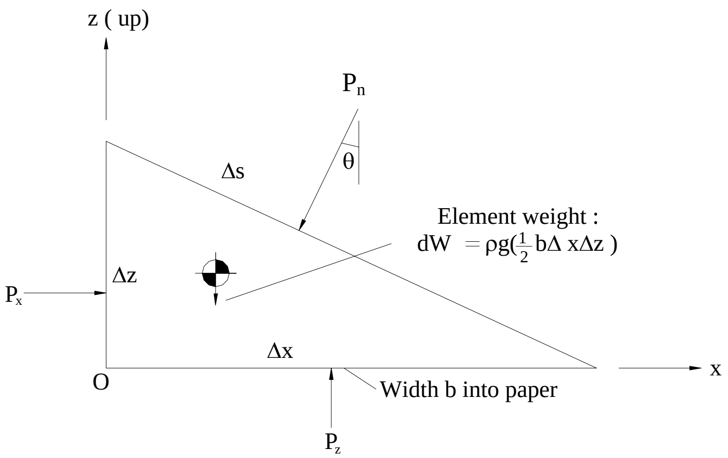


Fig. 2.1

$$\sum F_x = p_x b\Delta z - p_n b\Delta s Sin\theta = \frac{\Delta x\Delta z}{2}\rho a_x = 0$$

$$\sum F_z = p_z b\Delta x - p_n b\Delta s Sin\theta - \frac{1}{2}\gamma b\Delta x\Delta z = \frac{\Delta x\Delta z}{2}\rho a_z = 0 \qquad (2.1)$$

In which p_x, p_z, p_n are the average pressures on the three faces, γ is the specific weight of the fluid, ρ is the specific mass, and a_x and a_z are the acceleration components of the wedge in the x and z direction respectively. The geometry of the wedge is such that

$$\Delta z = \Delta s Sin\theta \quad , \ \Delta x = \Delta s Cos\theta$$

Substitution into Equ. (2.1) and rearrangement give

$$p_x = p_n \quad , \ p_z = p_n + \frac{1}{2}\gamma\Delta z \tag{2.2}$$

These relations illustrate two important principles of the hydrostatic, or shear free, condition:

1) There is no pressure change in the horizontal direction,
2) There is a vertical change in pressure proportional to the specific mass, gravity and depth change.

In the limit as the fluid wedge shrinks to a point, $\Delta z \to 0$ and Equ. (2.2) becomes

$$p_x = p_z = p_n = p \tag{2.3}$$

Since θ is arbitrary, we conclude that the pressure p at a point in a static fluid is independent of orientation, and has the same value in all directions.

2.2. PRESSURE VARIATION IN A STATIC FLUID

The fundamental equation of fluid statics is that relating pressure, specific mass and vertical distance in a fluid. This equation may be derived by considering the static equilibrium of a typical differential element of fluid (Fig. 2.2). The z-axis is in the direction parallel to the gravitational force field (vertical). Applying Newton's first law ($\Sigma F_x = 0$ and $\Sigma F_z = 0$) to the element

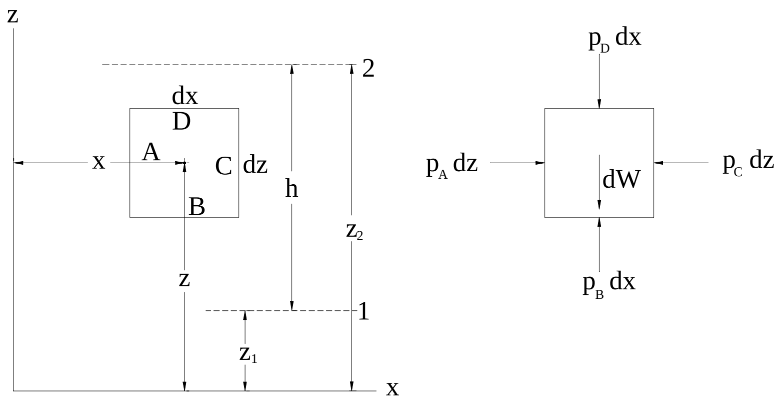


Fig. 2.2

And using the average pressure on each face to closely approximate the actual pressure distribution on the differential element (recall dx and dz are very small), give

$$\sum F_x = p_A dz - p_C dz = 0$$

$$\sum F_z = p_B dx - p_D dx - dW = 0 \qquad (2.4)$$

In which p and γ are functions of x and z. In partial derivation notation the pressures on the faces of the element are, in terms of pressure p in the center

$$p_A = p - \frac{\partial p}{\partial x}\frac{dx}{2} \quad , \; p_B = p - \frac{\partial p}{\partial z}\frac{dz}{2}$$

$$p_C = p + \frac{\partial p}{\partial x}\frac{dx}{2} \quad , \; p_D = p + \frac{\partial p}{\partial z}\frac{dz}{2}$$

The weight of the small element is dW = γdxdz (as dx and dz approach zero in the limiting process for partial differentiation, any variations in γ over the element will vanish). Thus, Eqs. (2.4) become

$$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)dz - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)dz = -\frac{\partial p}{\partial x}dxdz = 0$$

And similarly

$$-\frac{\partial p}{\partial z}dzdx - \gamma dxdz = 0$$

Canceling the dxdz in both cases gives

$$\frac{\partial p}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = \frac{dp}{dz} = -\gamma \qquad (2.5)$$

The first of these equations shows there is no variation of pressure with horizontal distance, that is, pressure is constant in a horizontal plane in a static fluid; therefore pressure is a function of z only and the total derivative may replace the partial derivative in the second equation, which is the basic equation of fluid statics.

Equ. (2.5) may be integrated directly to find

$$z_2 - z_1 = \int_{p_2}^{p_1} \frac{dp}{\gamma} \qquad (2.6)$$

For a fluid of constant specific weight, the integration yields

$$z_2 - z_1 = h = \frac{p_1 - p_2}{\gamma} \quad \text{or} \quad p_1 - p_2 = \gamma(z_2 - z_1) = \gamma h \tag{2.7}$$

Permitting ready calculation of the increase of pressure with depth in a fluid of constant specific weight. Equ. (2.7) also shows that pressure differences ($p_1 - p_2$) may be readily expressed as a *head h* of fluid of specific weight γ. Thus pressures are often quoted as heads in millimeters of mercury, meters of water. The open *manometer* and *piezometer columns* of Fig. 2.3 illustrate the relation of pressure to head.

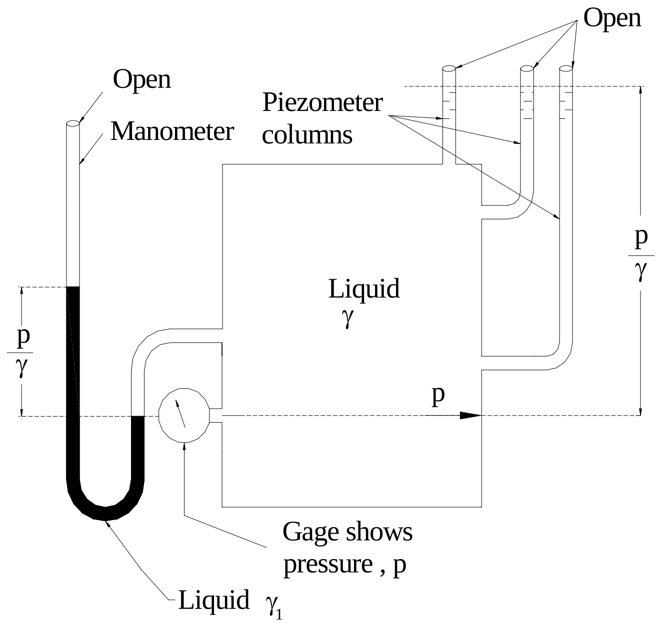


Fig. 2.3

Equ. (2.7) may be arranged fruitfully to

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 = \text{Constant} \tag{2.8}$$

for later comparison with equations of fluid flow. Taking points 1 and 2 as typical, it is evident from Equ. (2.8) that the quantity (z + p/γ) is the same for all points in a static fluid. This may be visualized geometrically as shown on Fig. 2.4.

Frequently, in engineering problems the liquid surface is exposed to atmospheric pressure; if the latter is taken to be zero, the dashed line of Fig. 2.4 will necessarily coincide with the liquid surface.

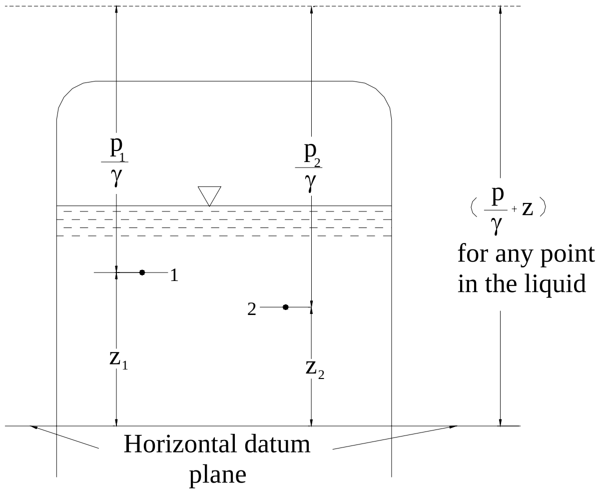


Fig. 2.4

2.3. THE HYDROSTATIC PARADOX

From Equ. (2.7) it can be seen that the pressure exerted by a fluid is dependent only on the vertical head of fluid and its specific weight; it is not affected by the weight of the fluid present. Thus, in Fig. 2.5 the four vessels all have the same base area A and filled to the same height with the same liquid of specific weight γ.

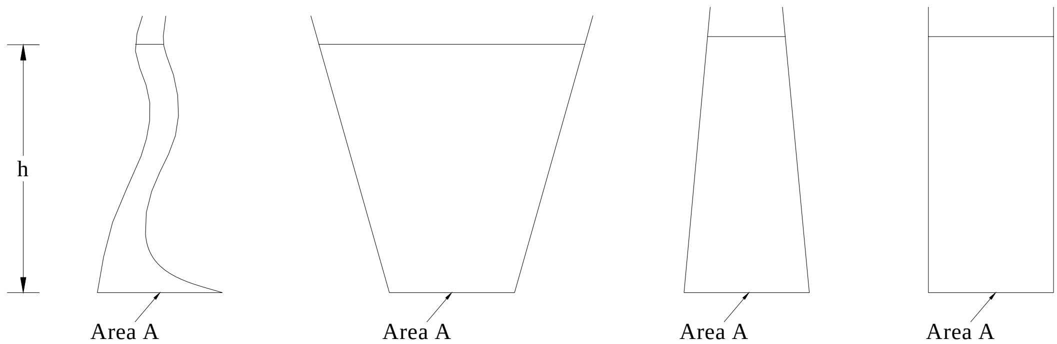


Fig. 2.5

Pressure on bottom in each case = $p = \gamma\times h$

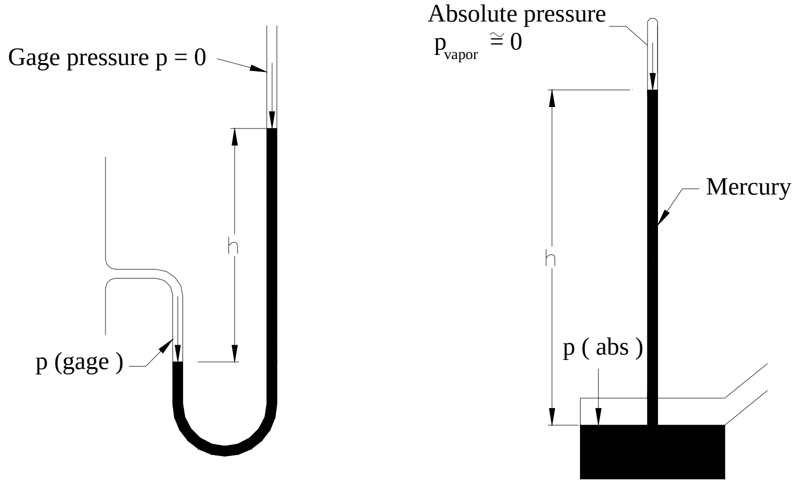
Force on bottom = Pressure×Area = $p\times A = \gamma\times h\times A$

Thus, although the weight of fluid is obviously different in the four cases, the force on the bases of the vessels is the same, depending on the depth h and the base area A.

2.4. ABSOLUTE AND GAGE PRESSURES

Pressures are measured and quoted in two different systems, one *relative (gage)*, and the other *absolute*; no confusion results if the relation between the systems and the common methods of measurement is completely understood.

Liquid devices that measure gage and absolute pressures are shown on Fig. 2.6; these are the open U-tube and conventional mercury barometer. With the U-tube open, atmospheric pressure will act on the upper liquid surface; if this pressure is taken to be zero, the applied gage pressure p will equal γh and h will thus be a direct measure of gage pressure.



(a) For gage pressure (b) For absolute pressure

Fig. 2.6

The mercury barometer (invented by Toricelli, 1643) is constructed by filling the tube with air-free mercury and inverting it with its open end beneath the mercury surface in the receptacle. Ignoring the small pressure of the mercury vapor, the pressure in the space above the mercury will be absolute zero and again $p = \gamma h$; here the height h is direct measure of the absolute pressure, p.

From the foregoing descriptions an equation relating (gage) and absolute pressures may now be written,

$$\text{Absolute pressure} = \text{Atmospheric pressure} \begin{matrix} -\text{Vacuum} \\ \\ +\text{Gage pressure} \end{matrix} \qquad (2.9)$$

Which allows easy conversion from one system to the other. Possibly a better picture of these relationships can be gained from a diagram such as that of Fig. 2.7 in which are shown two typical pressures, A and B, one above, the other below, atmospheric pressure, with all the relationships indicated graphically.

At sea level standard atmosphere $p_{atm} = p_0 = 10.33$ t/m^2, a pieozometer column of mercury will stand at a height of 0.76 m. However, if water were used, a reading of about 10.33 m would be obtained.

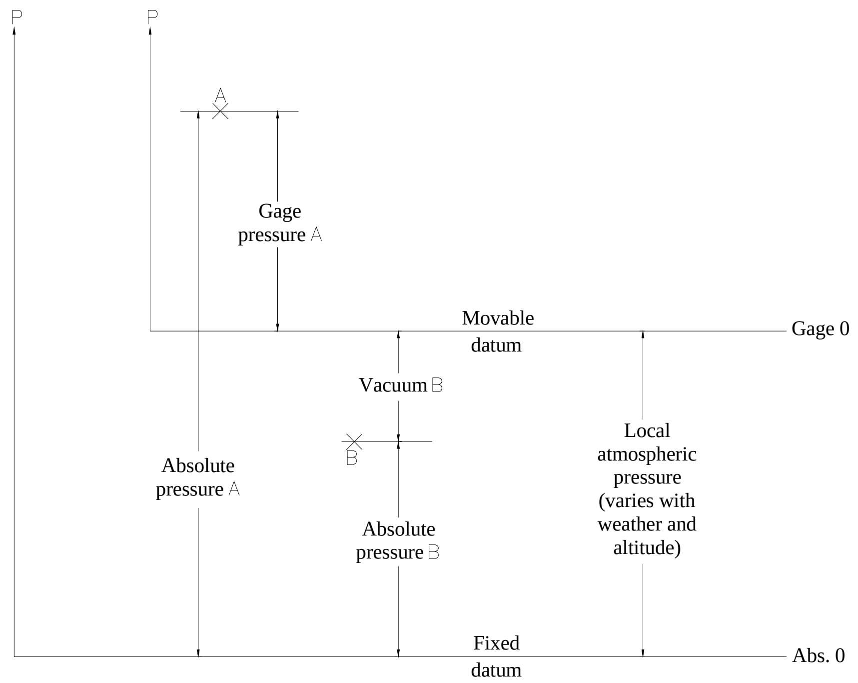


Fig. 2.7

EXAMPLE 2.1:A cylinder contains a fluid at a relative (gage) pressure of 35 t/m^2. Express this pressure in terms of a head of, a) water (γ_{water} = 1000 kg/m^3), b) mercury (γ_{Hg} = 13.6 t/m^3).

What would be the absolute pressure in the cylinder if the atmospheric pressure is 10.33 t/m^2?

SOLUTION:

From Equ. (2.7), head, h = p/γ.

a) Putting p = 35 t/m^2, γ = 1 t/m^3,

Equivalent head of water = $\dfrac{35}{1} = 35m.$

b) For mercury γ_{Hg} = 13.6 t/m^3,

Equivalent head of mercury = $\dfrac{35}{13.6} = 2.57m.$

Absolute pressure = Gage pressure + Atmospheric pressure

$p_{abs} = 35 + 10.33 = 45.33\,t/m^2$

2.5. MANOMETER

From the hydrostatic Equ. (2.7), a change in elevation $(z_2 - z_1)$ of a liquid is equivalent to a change in pressure $(p_2 - p_1)/\gamma$. Thus a static column of one or more liquids can be used to measure differences between two points. Such a device is called a *manometer*. If multiple fluids are used, we must change the specific weight in the equation as move from one fluid to another. Fig. 2.8 illustrates the use of the equation with a column of multiple fluids. The pressure change through each fluid is calculated separately. If we wish to know the total change $(p_5 - p_1)$, we add successive changes $(p_2 - p_1)$, $(p_3 - p_2)$, $(p_4 - p_3)$, and $(p_5 - p_4)$. The intermediate values of p cancel, and we have, for the example of Fig. 2.8,

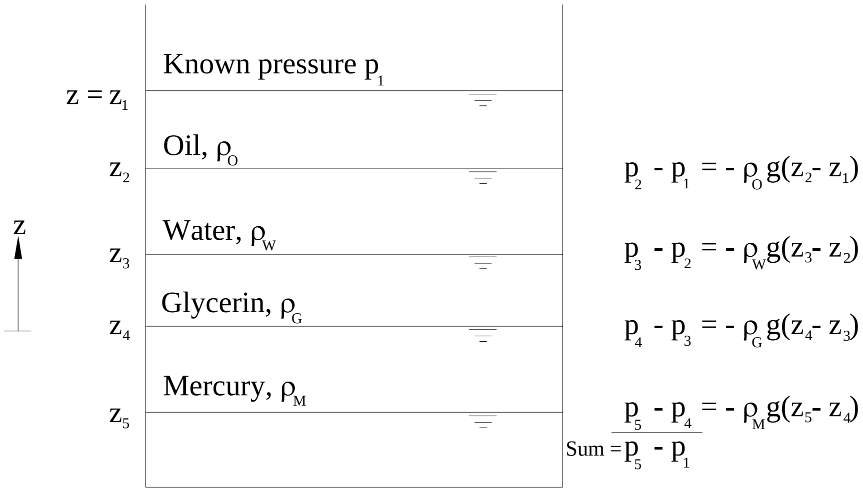


Fig. 2.8

$$p_5 - p_1 = -\gamma_o(z_2 - z_1) - \gamma_w(z_3 - z_2) - \gamma_G(z_4 - z_3) - \gamma_{Hg}(z_5 - z_4) \qquad (2.10)$$

No additional simplification is possible on the right-hand side because of the different specific weights. Notice that we have placed the fluids in order from the lightest on top to the heaviest at bottom.

When calculating hydrostatic pressure changes, engineers work instinctively by simply having the pressure increase downward and decrease upward.

$$p_{down} = p^{up} + \gamma|\Delta z| \qquad (2.11)$$

Thus, without worrying too much about which point is z_1 and which is z_2, the equation simply increases or decreases the pressure according to whether one is moving down or up. For example, Equ. (2.10) could be written in the following “multiple increase” mode:

$$p_5 = p_1 + \gamma_o|z_1 - z_2| + \gamma_w|z_2 - z_3| + \gamma_G|z_3 - z_4| + \gamma_{Hg}|z_4 - z_5|$$

That is, keep adding on pressure increments as you move down through the layered fluid. A different application is a manometer, which involves both “up” and “down” calculations.

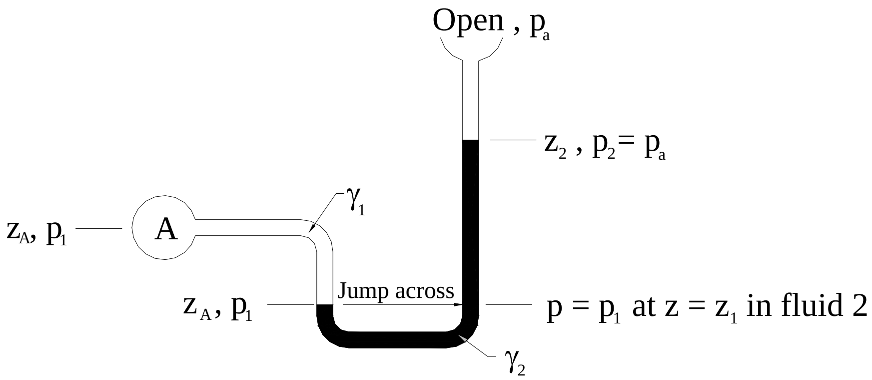


Fig. 2.9

Fig. 2.9 shows a simple manometer for measuring p_A in a closed chamber relative to atmospheric pressure p_0, in other words, measuring gage (relative) pressure. The chamber fluid γ_1 is combined with a second fluid γ_2, perhaps for two reasons: 1) To protect the environment from a corrosive chamber fluid or, 2) Because a heavier fluid γ_2 will keep z_2 small and the open tube can be shorter. One can apply the basic hydrostatic Equ. (2.7). Or, more simply, one can begin at A, apply Equ. (2.11) "down" to z_1, jump across fluid 2 (see Fig. 2.9) to the same pressure p_1, and then use Equ. (2.11) "up" to level z_2:

$$p_A + \gamma_1 \left| z_A - z_1 \right| - \gamma_2 \left| z_1 - z_2 \right| = p_2 = p_{atm} = p_0 \qquad (2.12)$$

The physical reason that we can "jump across" at section 1 in that a continuous length of the same fluid connects these two elevations. The hydrostatic relation (Equ. 2.7) requires this equality as a form of Pascal's law:

Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure.

This idea of jumping across to equal pressures facilitates multiple-fluid problems.

EXAMPLE 2.2: A U-tube manometer in Fig. 2.10 is used to measure the gage pressure of a fluid P of specific weight $\gamma_P = 800$ kg/m^3. If the specific weight of the liquid Q is $\gamma_Q = 13.6 \times 10^3$ kg/m^3, what will be the gage pressure at A if, a) $h_1 = 0.5$ m and D is 0.9 m above BC, b) $h_1 = 0.1$ m and D is 0.2 m below BC?

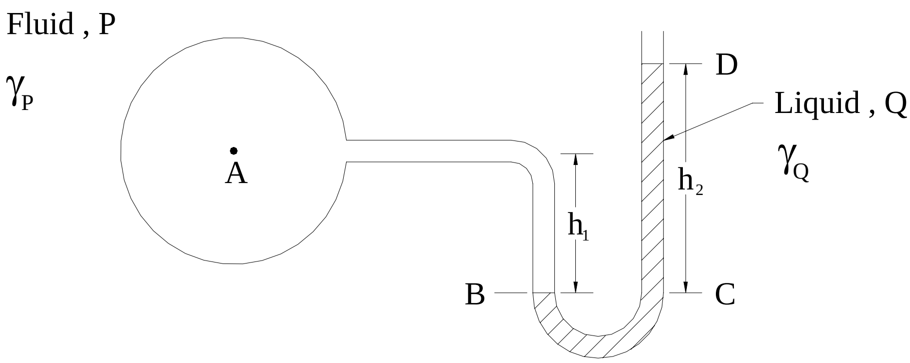


Fig. 2.10

SOLUTION:

a) In Equ. (2.12), γ_1 = 0.8 t/m^3, γ_2 = 13.6 t/m^3, ($z_A - z_1$) = 0.5 m, ($z_1 - z_2$) = 0.9 m.

$$p_A + 0.8\times 0.5 - 13.6\times 0.9 = p_0 = 0$$
$$p_A = 13.6\times 0.9 - 0.8\times 0.5 = 11.84\, t/m^2$$

b) Putting $|z_A - z_1|$ =0.1 m and $|z_1 - z_2|$ =-0.2 m into Equ. (2.12) gives,

$$p_A + 0.8\times 0.1 - 13.6\times(-0.2) = p_0 = 0$$
$$p_A = -0.08 - 2.72 = -2.8\, t/m^2$$

The negative sign indicating that p_A is below atmospheric pressure. The absolute pressure at A is according to Equ. (2.9),

$$p_{A_{abs}} = p_0 + p_A = 10.33 - 2.8 = 7.53\, t/m^2$$

Fig. 2.11 illustrates a multiple-fluid manometer problem for finding the difference in pressure between two chambers A and B. We repeatedly apply Equ. (2.7) jumping across at equal pressures when we come to a continuous mass of the same fluid. Thus, in Fig. 2.11, we compute four pressure differences while making three jumps:

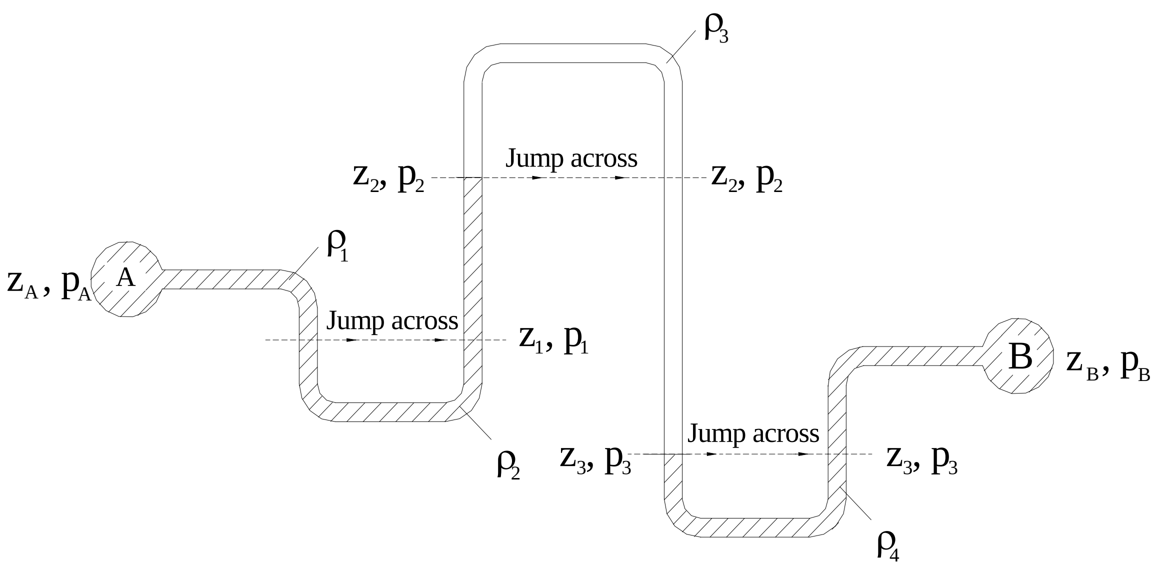


Fig. 2.11

$$\begin{aligned} p_A - p_B &= (p_A - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_B) \\ &= -\gamma_1(z_A - z_1) - \gamma_2(z_1 - z_2) - \gamma_3(z_2 - z_3) - \gamma_4(z_3 - z_B) \end{aligned} \qquad (2.13)$$

The intermediate pressures $p_{1,2,3}$ cancel. It looks complicated, but it is merely sequential. One starts at A, goes down to 1, jump across, goes down to 3, jumps across, and finally goes up to B.

EXAMPLE 2.3: Pressure gage B is to measure the pressure at point A in a water flow. If the pressure at B is 9 t/m^2, estimate the pressure at A. γ_{water} = 1000 kg/m^3, γ_{Hg} = 13600 kg/m^3, γ_{oil}= 900 kg/m^3.

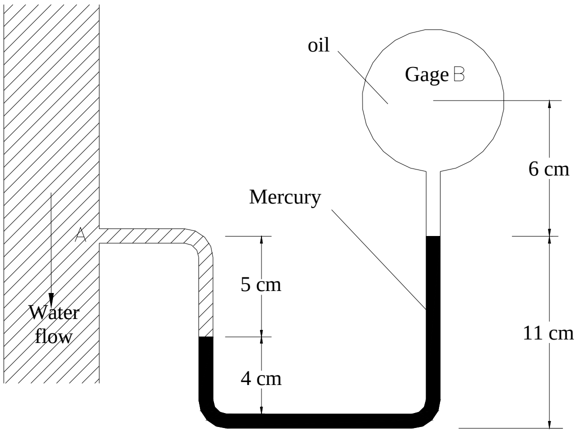


Fig. 2.12

SOLUTION: Proceed from A to B, calculating the pressure change in each fluid and adding:

$$p_A - \gamma_w (\Delta z)_w - \gamma_{Hg} (\Delta z)_{Hg} - \gamma_o (\Delta z)_o = p_B$$

or

$$p_A - 1000 \times (-0.05) - 13600 \times 0.07 - 900 \times 0.06$$

$$= p_A + 50 - 952 - 54 = p_B = 9000\, kg/m^2$$

$$p_A = 9956\, kg/m^2 \cong 9.96\, t/m^2$$

2.6. FORCES ON SUBMERGED PLANE SURFACES

The calculation of the magnitude, direction, and location of the total forces on surfaces submerged in a liquid is essential in the design of dams, bulkheads, gates, ships, and the like.

For a submerged plane, horizontal area the calculation of these force properties is simple because the pressure does not vary over the area; for nonhorizontal planes the problem is complicated by pressure variation. Pressure in constant specific weight liquids has been shown to vary linearly with depth (Equ. 2.7), producing the typical pressure distributions and resultant forces on the walls of a container of Fig. 2.13.

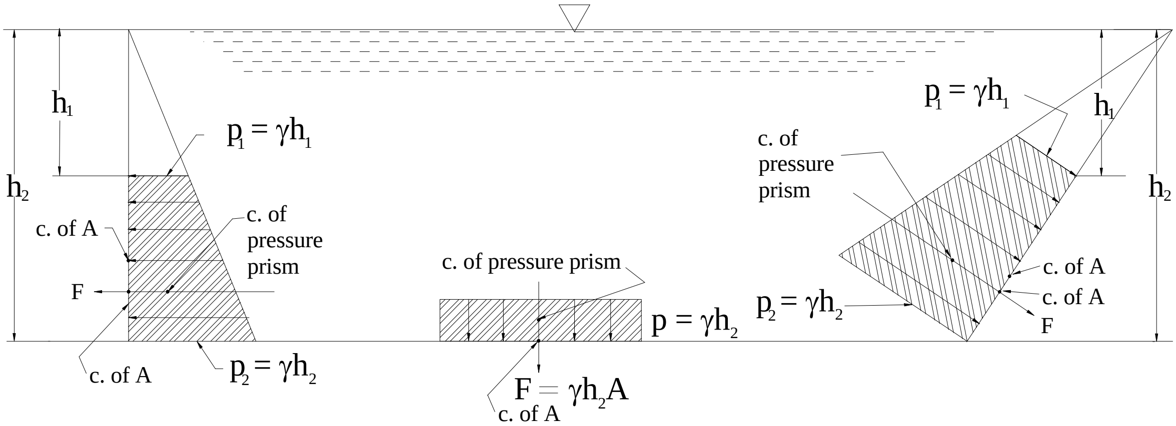


Fig. 2.13

The shaded areas, appearing as trapezoids are really volumes, known as *pressure prisms*. In mechanics it has been shown that the resultant force, F, is equal to the volume of the pressure prism and passes through its centroid.

Now consider the general case of a plane submerged area, A’ B’, such that of Fig. 2.14. It is inclined θ^0 from the horizontal.

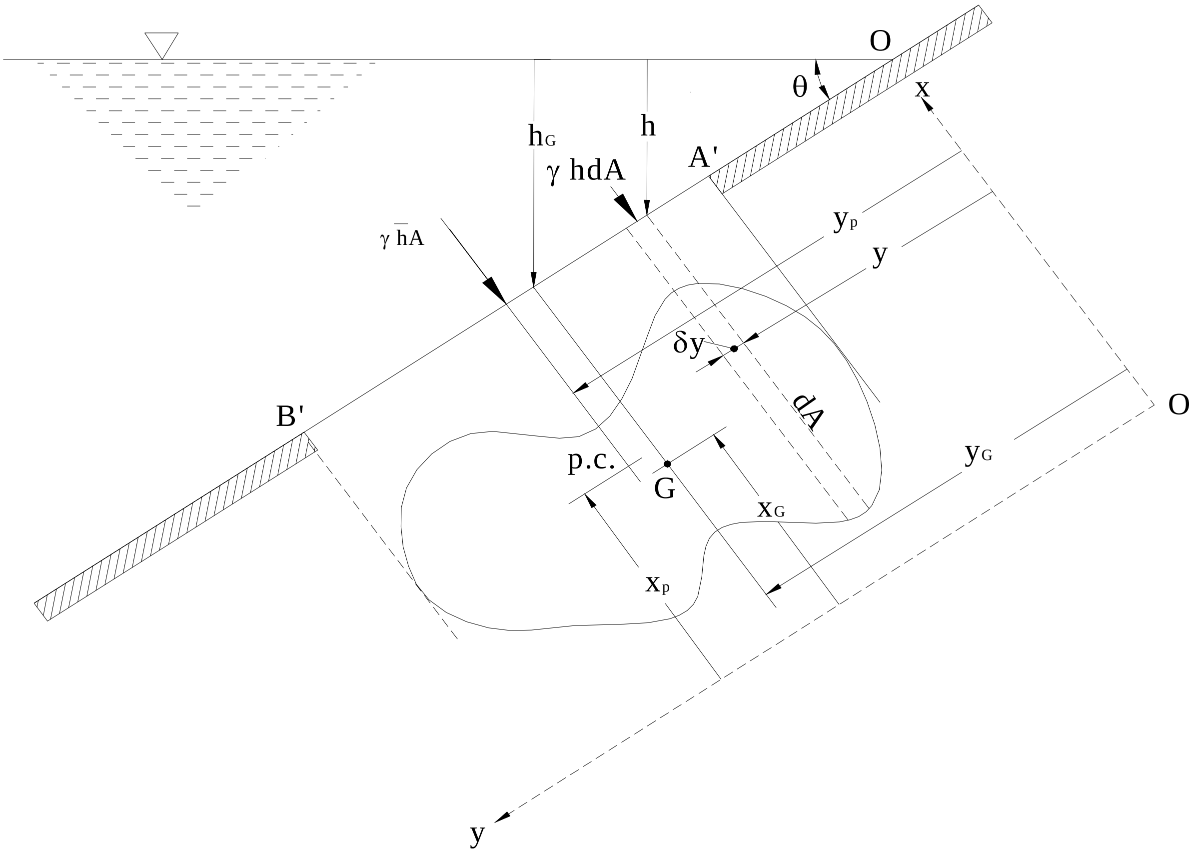


Fig. 2.14

The intersection of the plane of the area and the free surface is taken as the x-axis. The y-axis is taken in the plane of the area, with origin O, as shown, in the free surface. The xy-plane portrays the arbitrary inclined area. The magnitude, direction, and line of action of the resultant force due to liquid, acting on one side of the area are sought.

The force, dF, on the area, dA, is given by,

$$dF = pdA = \gamma h dA = \gamma y Sin\theta dA \qquad (2.14)$$

The integral over the area yields the magnitude of force F, acting on one side of the area,

$$F = \int_A pdA = \gamma Sin\theta \int_A ydA = \gamma Sin\theta y_G A = \gamma h_G A \qquad (2.15)$$

With the relations from Fig. 2.14, $y_G \times Sin\theta = h_G$, and $p_G = \gamma \times h_G$, the pressure at the centroid of the area. In words, the magnitude of force exerted on one side of the plane area submerged in a liquid is the product of the area and the pressure at its centroid. As all force elements are normal to the surface, the line of action of the resultant is also normal to the surface.

The line of action of the resultant force has its piercing point in the surface at a point called the *pressure center,* with coordinates (x_P,y_P). Unlike that for the horizontal surface, the center of pressure of an inclined surface is not at the centroid. To find the pressure center, the moments of the resultant $x_P \times F$, $y_p \times F$ are equated to the moment of the forces about the y-axis and x-axis, respectively; thus

$$x_P F = \int_A x dF = \int_A x p dA \tag{2.16}$$

$$y_P F = \int_A y dF = \int_A y p dA \tag{2.17}$$

After solving for the coordinates of pressure center,

$$x_P = \frac{1}{F} \int_A x p dA \tag{2.18}$$

$$y_P = \frac{1}{F} \int_A y p dA \tag{2.19}$$

Eqs. (2.18) and (2.19) may be transformed into general formulas as follows:

$$x_P = \frac{1}{\gamma y_G A Sin\theta} \int_A x \gamma y Sin\theta dA = \frac{1}{y_G A} \int_A x y dA = \frac{I_{xy}}{y_G A} \tag{2.20}$$

Since the products of inertia $\bar{I}_{xy}$ about centroidal axes parallel to the xy-axes produces,

$$I_{xy} = \bar{I}_{xy} + x_G y_G A \tag{2.21}$$

Equ. (2.20) takes the form of,

$$x_P = \frac{\bar{I}_{xy}}{y_G A} + x_G \tag{2.22}$$

When either of the centroidal axes, $x = x_G$ or $y = y_G$, is an axis of symmetry for the surface, $\bar{I}_{xy}$ vanishes and pressure center lies on $x = x_G$. Since $\bar{I}_{xy}$ may be either positive or negative, the pressure center may lie on either side of the line $x = x_G$. To determine y_P by formula, with Eqs. (2.14) and (2.19)

$$y_P = \frac{1}{\gamma y_G A Sin\theta}\int_A y\gamma y Sin\theta dA = \frac{1}{y_G A}\int_A y^2 dA = \frac{I_x}{y_G A} \tag{2.23}$$

In the parallel-axis theorem for moments of inertia

$$I_x = I_G + y_G^2 A \tag{2.24}$$

If I_x is eliminated from Equ. (2.23)

$$y_P = \frac{I_G}{y_G A} + y_G \tag{2.25}$$

or

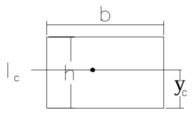
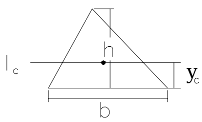
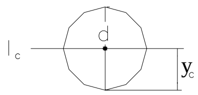
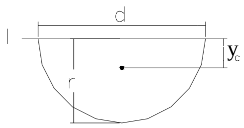
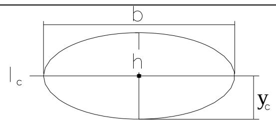
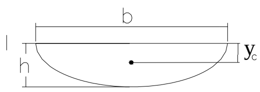
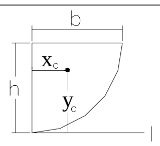
$$y_P - y_G = \frac{I_G}{y_G A} \tag{2.26}$$

I_G is always positive; hence (y_P – y_G) is always positive, and the pressure center is always below the centroid of the surface. It should be emphasized that y_G and (y_P–y_G) are distances in the plane of the surface. A summary of I_G's for common areas is given in Table 2.1.

The areas of irregular forms may be divided into simple areas, the forces being located on them, and the location of their resultant being found by the methods of statics. The point where the line of action of the resultant force intersects the area is the center of pressure for the composite area.

Table 2.1

Properties of Areas

	Sketch	Area	Location of Centroid	I or I_c
Rectangle		bh	$y_c = \frac{h}{2}$	$I_c = \frac{bh^3}{12}$
Triangle		$\frac{bh}{2}$	$y_c = \frac{h}{3}$	$I_c = \frac{bh^3}{36}$
Circle		$\frac{\pi d^2}{4}$	$y_c = \frac{d}{2}$	$I_c = \frac{\pi d^4}{64}$
Semicircle		$\frac{\pi d^2}{8}$	$y_c = \frac{4r}{3\pi}$	$I_c = \frac{\pi d^4}{128}$
Ellipse		$\frac{\pi bd}{4}$	$y_c = \frac{h}{2}$	$I_c = \frac{\pi bh^3}{64}$
Semiellipse		$\frac{\pi bd}{4}$	$y_c = \frac{4h}{3\pi}$	$I_c = \frac{\pi bh^3}{16}$
Parabola		$\frac{2}{2}bh$	$y_c = \frac{3h}{5}$ $x_c = \frac{3b}{8}$	$I_c = \frac{2bh^3}{7}$

EXAMPLE 2.4: The gate in Fig. 2.15 is 5 m wide, is hinged at point B, and rests against a smooth wall at point A. Compute,

a) The force on the gate due to the water pressure,
b) The horizontal force P exerted by the wall at point A,
c) The reactions at hinge B.

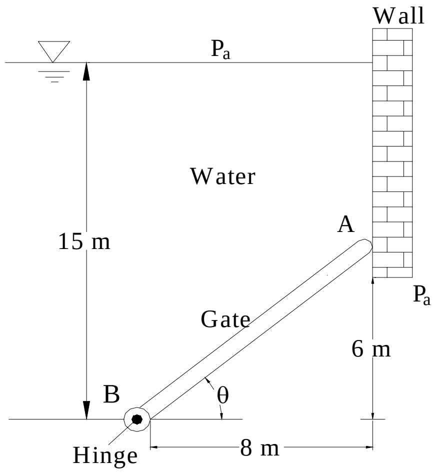


Fig. 2.15

SOLUTION:

a) By geometry the gate is 10 m long from A to B, and its centroid is halfway between, or at elevation 3 m above point B. The depth h_C is thus 15 – 3 = 12 m. The gate area is 5×10 = 50 m^2. Neglect p_0 (atmospheric pressure) as acting on both sides of the gate. From Equ. (2.15) the hydrostatic force on the gate is

$$F = p_C A = \gamma h_C A = 1 \times 12 \times 50 = 600 ton$$

b) First we must find the center of pressure of F. A free-body diagram of the gate is shown in Fig. 2.16. The gate is rectangle, hence

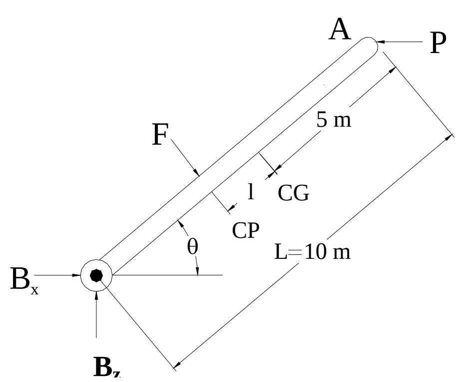


Fig. 2.16

$$I_{xy} = 0 \qquad \text{and} \qquad I_G = \frac{bL^3}{12} = \frac{5 \times 10^3}{12} = 417 m^4$$

The distance l from the CG to the CP is given by Equ. (2.26) since p_0 is neglected.

$$Sin\theta = \frac{6}{10} = 0.6 \quad , \ \theta = 37^0$$

$$y_G = \frac{h_G}{Sin\theta} = \frac{12}{0.6} = 20m$$
$$l = y_P - y_G = \frac{I_G}{y_G A} = \frac{417}{20\times 50} = 0.417m$$

The distance from point B to force F is thus 10 – l –5 = 10 – 0.417 – 5 =4.583 m. Summing moments counterclockwise about B gives

$$PLSin\theta - F(5-l) = 0$$
$$P = \frac{F(5-l)}{LSin\theta} = \frac{600\times(5-0.417)}{10\times 0.6} = 458.3ton$$

c) With F and P known, the reactions B_x and B_z are found by summing forces on the gate.

$$\sum F_x = B_x + FSin\theta - P = 0$$
$$B_x = 458.3 - 600\times 0.6 = 98.3ton$$

$$\sum F_z = B_z - FCos\theta = 0$$
$$B_z = 600\times 0.8 = 480ton$$

2.7. DIFFERENT PRESSURES ON TWO SIDES

The pressure has been considered as varying from zero at M to NK at N. In reality there is some pressure on the surface of the liquid, which might be represented by an equivalent height MO, and the absolute pressure on the left-hand side of the plane will vary as shown by ODE (Fig. 2.17)

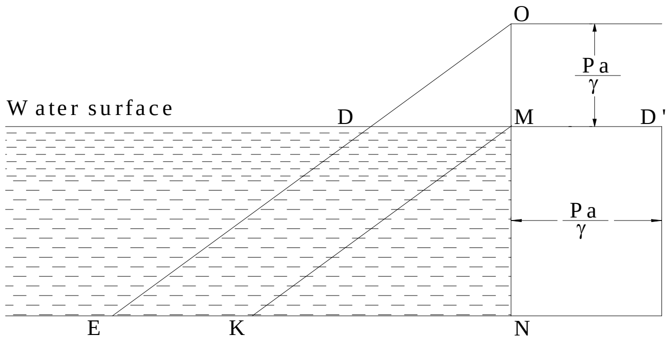


Fig. 2.17

In most practical cases it is the difference between the forces on the two sides is desired. The pressure of the air upon the surface of the liquid also produces a uniform pressure over the right-hand side of the area and thus MO = MD = MD'; and as the same air pressure acts a like on both sides, it mat be neglected altogether.

In a case such as that in Fig. 2.18, where surface represented by the trace MN is submerged by a liquid at two different heights on the two sides, the pressure variations are represented by CDE and LK. If the liquids are of the same specific weight, triangles DEN and LNK are equal. Thus the net pressure difference on the two sides is the uniform value DL, which is equal to γh.

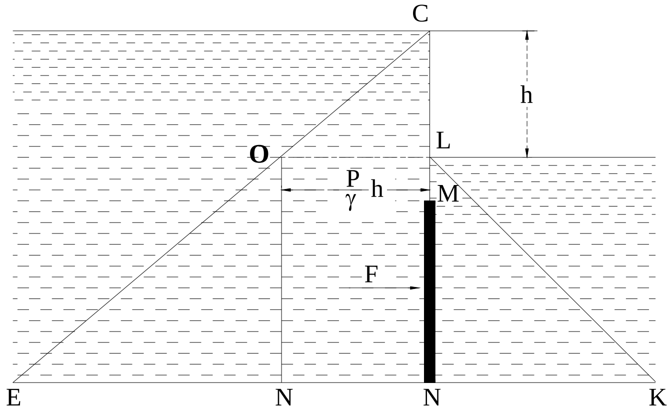


Fig. 2.18

Hence on any area by the same specific weight liquid on both sides but with a difference in level h as in Fig. (2.18), the resultant force is

$$F = \gamma h A \tag{2.27}$$

and it will be applied at the centroid of the area.

2.8.FORCE ON A CURVED SURFACE

If a surface is curved, it is convenient to calculate the horizontal and vertical components of the resultant force.

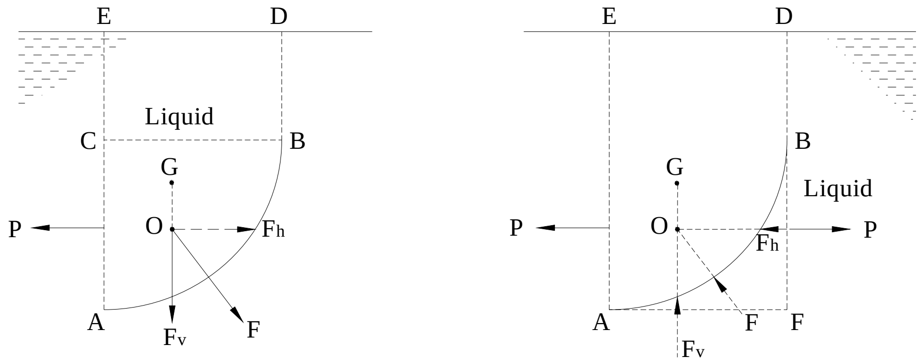


Fig. 2.19

In Fig. 2.19 (a) and (b), AB is the immersed surface and F_h and F_v are the horizontal and vertical components of the resultant force F of the liquid on one side of the surface. In Fig. 2.19. (a) the liquid lies above the immersed surface, while in Fig. 2.19 (b) it acts below the surface.

In Fig. 2.19 (a), if ACE is a vertical plane through A, and BC is a horizontal plane, then, since element ACB is in equilibrium, the resultant force F on AC must equal the horizontal component F_h of the force exerted by the fluid on AB because there are no other horizontal forces acting. But AC is the projection of AB on a vertical plane, therefore,

Horizontal component, F_h = Resultant force on the projection of AB on a vertical plane

Also, for equilibrium, P and F_h must act in the same straight line; therefore, the horizontal component F_h acts through the center of pressure of the projection of AB on a vertical plane.

Similarly, in Fig. 2.19 (b), element ABF is in equilibrium, and the horizontal component F_h is equal to the resultant force on the projection BF of the curved surface AB on a vertical plane, and acts through the center of pressure of this projection.

In Fig. 2.19 (a), the vertical component F_v will be entirely due to the weight of the fluid in the area ABDE lying vertically above AB. There are no other vertical forces, since there can be no shear forces on AE and BD because the fluid is at rest. Thus,

Vertical component, F_v = Weight of fluid vertically above AB

and will act vertically downwards through the center of gravity G of ABDE.

In Fig. 2.19 (b), if the liquid is on the right side of the surface AB, this liquid would be in equilibrium under its own weight and the vertical force on the boundary AB. Therefore,

Vertical component, F_v = Weight of the volume of the same fluid which would lie vertically above AB

and will act vertically upwards through the center of gravity G of this imaginary volume of fluid.

The resultant force F is found by combining the components vertically. If the surface is of uniform with perpendicular to the diagram, F_h and F_v will intersect at O. Thus,

Resultant force, $F = \sqrt{F_h^2 + F_v^2}$

and acts through O at an angle θ given by $Tan\theta = F_v / F_h$.

In the special case of a cylindrical surface, all the forces on each small element of area acting normal to the surface will be radial and will pass through the center of the curvature O (Fig. 2.20). The resultant force F must, therefore, also pass through the center of curvature O.

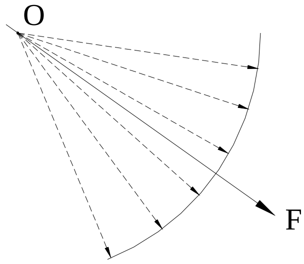


Fig. 2.20

EXAMPLE 2.5: A sluice gate is in the form of a circular arc of radius 6 m as shown in Fig. 2.21. Calculate the magnitude and direction of the resultant force on the gate, and the location with respect to O of a point of its line of action.

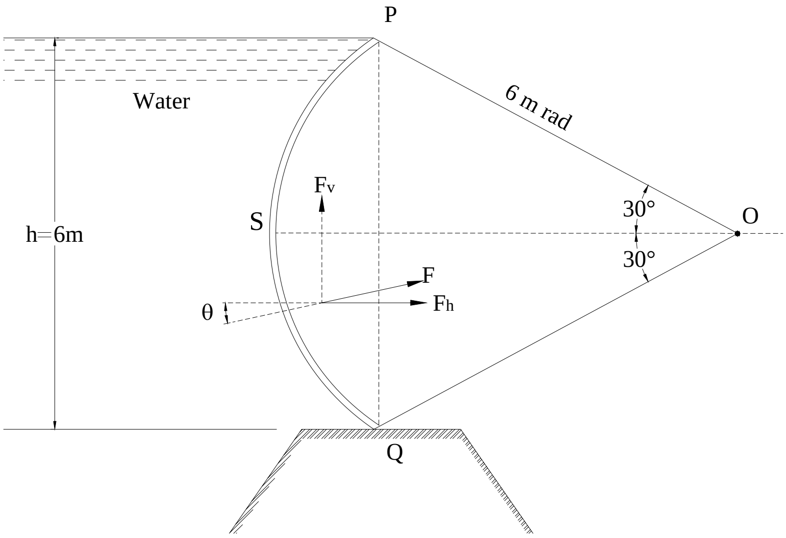


Fig. 2.21

SOLUTION:

Since the water reaches the top of the gate,

Depth of water, $h = 2 \times 6 \times Sin30^0 = 6m$

Horizontal component of force on gate = F_h (per unit length)

F_h = Resultant force on PQ per unit length

$$F_h = \gamma \times h \times \frac{h}{2} = \frac{\gamma h^2}{2}$$

$$F_h = \frac{1 \times 6^2}{2} = 18ton$$

Vertical component of force on gate = F_v (per unit length)

F_v = Weight of water displaced by segment PSQ

F_v = (Sector OPSQ – Triangle OPQ)×γ

$$F_v = \left(\frac{60}{360} \times \pi \times 6^2 - 6 \times Sin30^0 \times 6 \times Cos30^0 \right) \times 1$$
$$F_v = 3.26 ton$$

Resultant force on gate,

$$F = \sqrt{F_h^2 + F_v^2}$$
$$F = \sqrt{18^2 + 3.26^2} = 18.29 ton/m$$

$$Tan\theta = \frac{F_v}{F_h} = \frac{3.26}{18} = 0.18$$

$\theta = 10^0 27'$ to the horizontal

Since the surface of the gate is cylindrical, the resultant force F must pass through O.

2.9. BUOYANCY AND FLOTATION

The familiar laws of buoyancy (Archimedes' principle) and flotation are usually stated:

1) A body immersed in a fluid is buoyed up by a force equal to the weight of fluid displaced,
2) A floating body displaces its own weight of the fluid in which it floats.

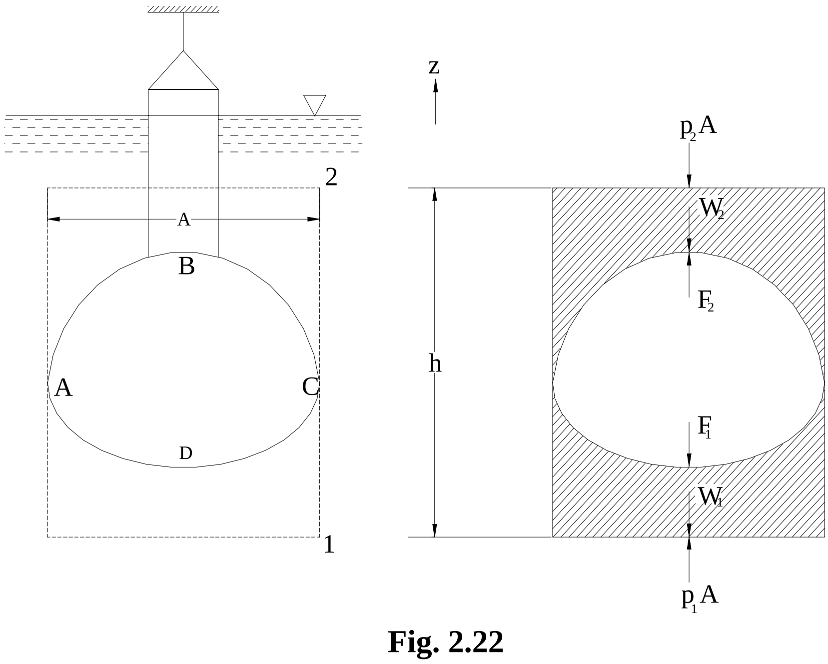


Fig. 2.22

A body ABCD suspended in a fluid of specific weight γ is illustrated in Fig. 2.22. Isolating a free body of fluid with vertical sides tangent to the body allows identification of the vertical forces exerted by the lower (ADC) and upper (ABC) surfaces of the body surrounding fluid. These are F_1 and F_2 with $(F_1 - F_2)$ the buoyant force on the body. For the upper portion of the free body

$$\sum F_z = F_2 - W_2 - p_2 A = 0$$

and for the lower portion

$$\sum F_z = F_1 + W_1 - p_1 A = 0$$

Whence (by subtracting of these equations)

$$F_B = F_1 - F_2 = (p_1 - p_2)A - (W_1 + W_2)$$

However, $(p_1 - p_2) = \gamma h$ and γhA is the weight of a cylinder of fluid extending between horizontal planes 1 and 2, and the right side of the equation for F_B is identified as the weight of volume of fluid exactly equal to the of the body

$$F_B = \gamma \times (\text{Volume of object}) \tag{2.28}$$

For the floating object of Fig. 2.23 a similar analysis will show that

$$F_B = \gamma \times (\text{Volume displaced}) \tag{2.29}$$

and, from static equilibrium of the object, its weight must be equal to this buoyant force; thus the object displaces its own weight of the liquid in which it floats.

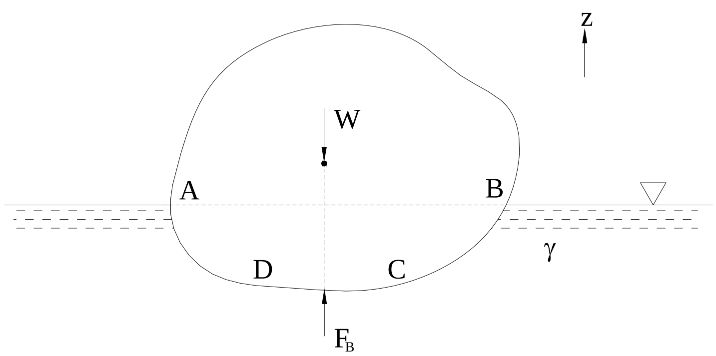


Fig. 2.23

EXAMPLE 2.6: A block of concrete weighs 100 kg in air and weighs 60 kg when immersed in water. What is the average specific weight of the block?

SOLUTION: The buoyant force is,

$$\sum F_z = 60 + F_B - 100 = 0$$
$$F_B = 40kg = \gamma_{water} \times (\text{Volume of the block})$$

$$V = \frac{40}{1000} = 0.04m^3$$

Therefore the specific weight of the block is,

$$\gamma = \frac{100}{0.04} = 2500\, kg/m^3 = 2.5\, ton/m^3$$

2.10. FLUIDS IN RELATIVE EQUILIBRIUM

If a fluid is contained in a vessel which is at rest, or moving with constant linear velocity, it is not affected by the motion of the vessel; but if the container is given a continuous acceleration, this will be transmitted to the fluid and affect the pressure distribution in it. Since the fluid remains at rest relative to the container, there is no relative motion of the particles of the fluid and, therefore, no shear stresses, fluid pressure being everywhere normal to the surface on which it acts. Under these conditions the fluid is said to be in relative equilibrium.

2.10.1. Pressure Distribution in a Liquid Subject to Horizontal Acceleration

Fig. 2.24 shows a liquid contained in a tank which has an acceleration a. A particle of mass m on the free surface at O will have the same acceleration as the tank and will be subjected to an accelerating force F. From Newton's second law,

$$F = ma \tag{2.30}$$

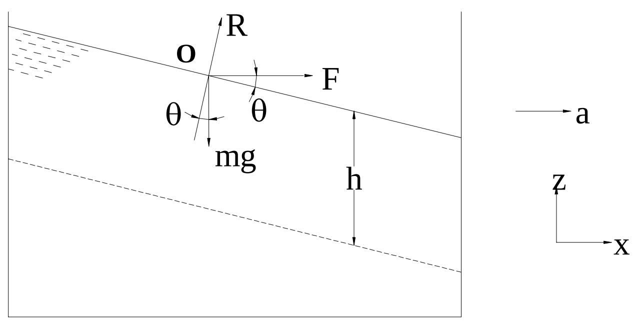


Fig. 2.24

Also, F is the resultant of the fluid pressure force R, acting normally to the free surface at O, and the weight of the particle mg, acting vertically. Therefore,

$$F = mgTan\theta \tag{2.31}$$

Comparing Eqs. (2.30) and (2.31)

$$Tan\theta = \frac{a}{g} \tag{2.32}$$

and is constant for all points on the free surface. Thus, the free surface is a plane inclined at a constant angle θ to the horizontal.

Since the acceleration is horizontal, vertical forces are not changed and the pressure at any depth h below the surface will be γh. Planes of equal pressure lie parallel to the free surface.

2.10.2. Effect of Vertical Acceleration

If the acceleration is vertical, the free surface will remain horizontal. Considering a vertical prism of cross-sectional area A (Fig. 2.25), subject to an upward acceleration a, then at depth h below the surface, where the pressure is p,

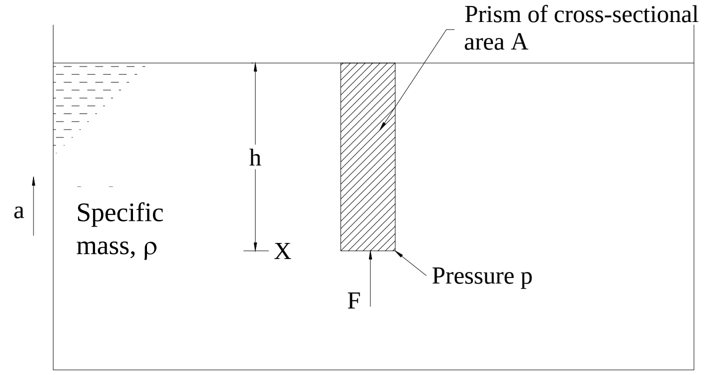


Fig. 2.25

Upward accelerating force, F = Force due to p – Weight of prism

$$F = pA - \gamma hA$$

By Newton's second law,

F = Mass of prism × Acceleration

$$F = \rho hA \times a$$

Therefore,

$$pA - \gamma hA = \rho hAa$$

$$p = \gamma h\left(1 + \frac{a}{g}\right) \qquad (2.33a)$$

If the acceleration a is downward towards to the center of the earth as gravitational acceleration, Equ. (2.33a) will take the form of,

$$p = \gamma h\left(1 - \frac{a}{g}\right) \qquad (2.33b)$$

2.10.3. General Expression for the Fluid in Relative Equilibrium

If ∂p/∂x, ∂p/∂y and ∂p/∂z are the rates of change of pressure p in the x, y and z directions (Fig. 2.26) and a_x, a_y and a_z the accelerations,

Force in x direction, $F_x = p\Delta y\Delta z - \left(p + \frac{\partial p}{\partial x}\Delta x \right)\Delta y\Delta z$

$$F_x = -\frac{\partial p}{\partial x}\Delta x\Delta y\Delta z$$

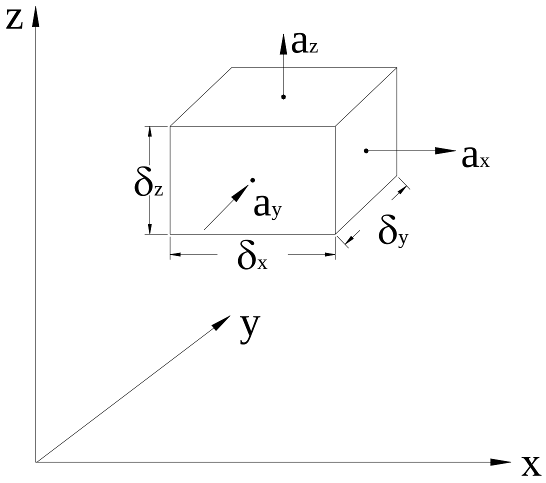


Fig. 2.26

By Newton's second law, $F_x = \rho\Delta x\Delta y\Delta z \times a_x$, therefore

$$-\frac{\partial p}{\partial x} = \rho a_x \tag{2.34}$$

Similarly, in the y direction

$$-\frac{\partial p}{\partial y} = \rho a_y \tag{2.35}$$

In the vertical z direction, the weight of the element $W = \rho g\Delta x\Delta y\Delta z$ must be considered:

$$F_z = p\Delta x\Delta y - \left(p + \frac{\partial p}{\partial z}\Delta z \right)\Delta x\Delta y - \rho g\Delta x\Delta y\Delta z$$

$$F_x = -\frac{\partial p}{\partial z}\Delta x\Delta y\Delta z - \rho g\Delta x\Delta y\Delta z$$

By Newton's second law, $F_z = \rho\Delta x\Delta y\Delta z \times a_z$, therefore,

$$-\frac{\partial p}{\partial z} = \rho\left(g + a_z\right) \tag{2.36}$$

For an acceleration a_s in any direction in the x-z plane making an angle ϕ with the horizontal, the components of the acceleration are

$$a_x = a_s Cos\phi \qquad \text{and} \qquad a_z = a_s Sin\phi$$

Now

$$\frac{dp}{ds} = \frac{\partial p}{\partial x}\frac{dx}{ds} + \frac{\partial p}{\partial z}\frac{dz}{ds} \tag{2.37}$$

For the free surface and all other planes of constant pressure, dp/ds = 0. If θ is the inclination of the planes of constant pressure to the horizontal, Tanθ = dz/dx. Putting dp/ds = 0 in Equ. (2.37)

$$\frac{\partial p}{\partial x}\frac{dx}{ds} + \frac{\partial p}{\partial z}\frac{dz}{ds} = 0$$

$$\frac{dz}{dx} = Tan\theta = -\frac{\partial p/\partial x}{\partial p/\partial z}$$

Substituting from Eqs. (2.34) and (2.36)

$$Tan\theta = -\frac{a_x}{g + a_z} \tag{2.38}$$

Or, in terms of a_s,

$$Tan\theta = -\frac{a_s Cos\phi}{\left(g + a_s Sin\phi\right)} \tag{2.39}$$

For the case of horizontal acceleration, ϕ = 0 and Equ. (2.39) gives Tanθ = $-a_s/g$, which agrees with Equ. (2.32). For vertical acceleration, $\phi = 90^0$ giving Tanθ = 0, indicating that the free surface remains horizontal.

EXAMPLE 2.7: A rectangular tank 1.2 m deep and 2 m long is used to convey water up a ramp inclined at an angle ϕ of 30^0 to the horizontal (Fig. 2.27).

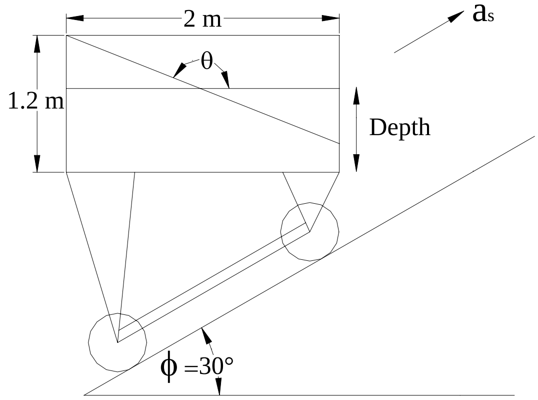


Fig. 2.27

Calculate the inclination of the water surface to the horizontal when,

a) The acceleration parallel to the slope on starting from bottom is 4 m/sec^2,
b) The deceleration parallel to the slope on reaching the top is 4.5 m/sec^2.

If no water is to be spilt during the journey what is the greatest depth of water permissible in the tank when it is at rest?

SOLUTION: The slope of the water surface is given by Equ. (2.39). During acceleration, a_s = 4 m/sec^2

$$Tan\theta_A = -\frac{a_s Cos\phi}{g + a_s Sin\phi} = -\frac{4\times Cos30^0}{9.81 + 4\times Sin30^0} = -0.2933$$

$$\theta_A = 163^0 39'$$

During retardation, a_s = - 4.5 m/sec^2,

$$Tan\theta_R = -\frac{(-4.5)\times Cos30^0}{9.81 - 4.5\times Sin30^0} = 0.5154$$

$$\theta_R = 27^0 16'$$

Since 180^0 - $\theta_R > \theta_A$, the worst case for spilling will be during retardation. When the water surface is inclined, the maximum depth at the tank wall will be

Depth + 0.5×Length×Tanθ

Which must not exceed 1.2 m if the water is not to be spilt. Putting length = 2 m, Tanθ = Tan θ_R = 0.5154,

Depth + 0.5×2×0.5154 = 1.2

Depth = 1.2 – 0.5154 = 0.6846 m

2.10.4. Forced Vortex

A body of fluid, contained in a vessel, which is rotating about a vertical axis with uniform angular velocity, will eventually reach relative equilibrium and rotate with the same angular velocity w as the vessel, forming a forced vortex. The acceleration of any particular of fluid at radius r due to rotation will be ($-w^2r$) perpendicular to the axis of rotation, taking the direction of r as positive outward from the axis. Thus, from Equ. (2.34),

$$\frac{dp}{dr} = -\rho w^2 r \qquad (2.40)$$

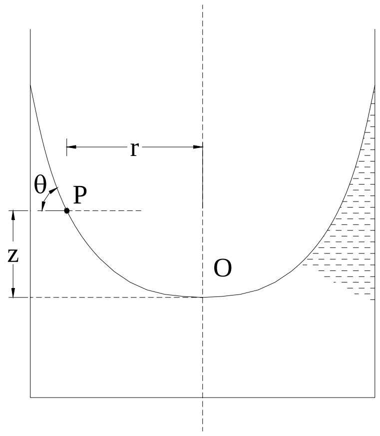


Fig. 2.28

Fig. 2.28 shows a cylindrical vessel containing liquid rotating about its axis, which is vertical. At any P on free surface, the inclination θ of the free surface is given by Equ. (2.38),

$$Tan\theta = -\frac{a_x}{g + a_z} = \frac{w^2 r}{g} = \frac{dz}{dr} \tag{2.41}$$

The inclination of the free surface varies with r and, if z is the height of P above O, the surface profile is given by integrating Equ. (2.41):

$$z = \int_0^x \frac{w^2 r}{g} dr = \frac{w^2 r^2}{2g} + C \tag{2.42}$$

Thus, the profile of the water is a paraboloid. Similarly, other surfaces of equal pressure will also be paraboloids.

The value of integration constant is found by specifying the pressure at one point. If z = 0 at point O, then the integration constant is zero. Then the Equ. (2.42) becomes,

$$z = \frac{w^2 r^2}{2g} \tag{2.43}$$

CHAPTER 3

KINEMATICS OF FLUIDS

3.1. FLUID IN MOTION

Fluid motion observed in nature, such as the flow of waters in rivers is usually rather chaotic. However, the motion of fluid must conform to the general principles of mechanics. Basic concepts of mechanics are the tools in the study of fluid motion.

Fluid, unlike solids, is composed of particles whose relative motions are not fixed from time to time. Each fluid particle has its own velocity and acceleration at any instant of time. They change both respects to time and space. For a complete description of fluid motion it is necessary to observe the motion of fluid particles at various points in space and at successive instants of time.

Two methods are generally used in describing fluid motion for mathematical analysis, the *Lagrangian* method and the *Eulerian* method.

The Lagrangian method describes the behavior of the individual fluid during its course of motion through space. In rectangular Cartesian coordinate system, Lagrange adopted a, b, c, and t as independent variables. The motion of fluid particle is completely specified if the following equations of motion in three rectangular coordinates are determined:

$$\begin{aligned} x &= F_1(a,b,c,t) \\ y &= F_2(a,b,c,t) \\ z &= F_3(a,b,c,t) \end{aligned} \tag{3.1}$$

Eqs. (3.1) describe the exact spatial position (x, y,z) of any fluid particle at different times in terms of its *initial position* ($x_0 = a$, $y_0 = b$, $z_0 = c$) at the given initial time $t = t_0$. They are usually referred to as parametric equations of the path of fluid particles. The attention here is focused on the paths of different fluid particles as time goes on. After the equations describing the paths of fluid particles are determined, the instantaneous velocity components and acceleration components at any instant of time can be determined in the usual manner by taking derivatives with respect to time.

$$u = \frac{dx}{dt} \quad , \; a_x = \frac{du}{dt} = \frac{d^2 x}{dt^t}$$

$$v = \frac{dy}{dt} \quad , \; a_y = \frac{dv}{dt} = \frac{d^2 y}{dt^2} \qquad (3.2)$$

$$w = \frac{dz}{dt} \quad , \; a_z = \frac{dw}{dt} = \frac{d^2 z}{dt^2}$$

In which u, v, and w, and a_x, a_y, and a_z are respectively the x, y, and z components of velocity and acceleration.

In the Eulerian method, the individual fluid particles are not identified. Instead, a fixed position in space is chosen, and the velocity of particles at this position as a function of time is sought. Mathematically, the velocity of particles at any point in the space can be written,

$$\begin{aligned} u &= f_1(x, y, z, t) \\ v &= f_2(x, y, z, t) \\ w &= f_3(x, y, z, t) \end{aligned} \qquad (3.3)$$

Euler chose x, y, z, and t as independent variables in his method.

The relationship between Eulerian and Lagrangian methods can be shown. According to the Lagrangian method, we have a set of Eqs. (3.2) for each particle which can be combined with Eqs. (3.3) as follows:

$$\begin{aligned} \frac{dx}{dt} &= u(x, y, z, t) \\ \frac{dy}{dt} &= v(x, y, z, t) \\ \frac{dz}{dt} &= w(x, y, z, t) \end{aligned} \qquad (3.4)$$

The integration of Eqs. (3.4) leads to three constants of integration, which can be considered as initial coordinates a, b, c of the fluid particle. Hence the solutions of Eqs. (3.4) give the equations of Lagrange (Eqs. 3.1).

Although the solution of Lagrangian equations yields the complete description of paths of fluid particles, the mathematical difficulty encountered in solving these equations makes the Lagrangian method impractical. In most fluid mechanics problems, knowledge of the behavior of each particle is not essential. Rather the general state of motion expressed in terms of velocity components of flow and the change of velocity with respect to time at various points in the flow field are of greater practical significance. Therefore the Eulerian method is generally adopted in fluid mechanics.

With the Eulerian concept of describing fluid motion, Eqs. (3.3) give a specific velocity field in which the velocity at every point is known. In using the velocity field, and noting that x, y, z are functions of time, we may establish the acceleration components a_x, a_y, and a_z by employing the chain rule of partial differentiation,

$$u = f_1(x, y, z, t) \quad , \quad a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

$$a_x = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) + \left(\frac{\partial u}{\partial t} \right)$$

$$v = f_2(x, y, z, t) \quad , \quad a_y = \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) + \left(\frac{\partial v}{\partial t} \right) \qquad (3.5)$$

$$w = f_3(x, y, z, t) \quad , \quad a_z = \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) + \left(\frac{\partial w}{\partial t} \right)$$

The acceleration of fluid particles in a flow field may be imagined as the superposition of two effects:

1) At a given time t, the field is assumed to become and remain steady. The particle, under such circumstances, is in the process of changing position in this steady field. It is thus undergoing a change in velocity because the velocity at various positions in this field will be different at any time t. This time rate of change of velocity due to changing position in the field is called *convective acceleration*, and is given the first parentheses in the preceding acceleration equations.
2) The term within the second parentheses in the acceleration equations does not arise from the change of particle, but rather from the rate of change of the velocity field itself at the position occupied by the particle at time t. It is called *local acceleration*.

3.2. UNIFORM FLOW AND STEADY FLOW

Conditions in a body of fluid can vary from point to point and, at any given point, can vary from one moment of time to the next. Flow is described as *uniform* if the velocity at a given instant is the same in magnitude and direction at every point in the fluid. If, at the given instant, the velocity changes from point to point, the flow is described as *non-uniform*.

A *steady* flow is one in which the velocity and pressure may vary from point to point but do not change with time. If, at a given point, conditions do change with time, the flow is described as *unsteady*.

For example, in the pipe of Fig. 3.1 leading from an infinite reservoir of fixed surface elevation, unsteady flow exits while the valve A is being opened or closed; with the valve opening fixed, steady flow occurs under the former condition, pressures, velocities, and the like, vary with time and location; under the latter they may vary only with location.

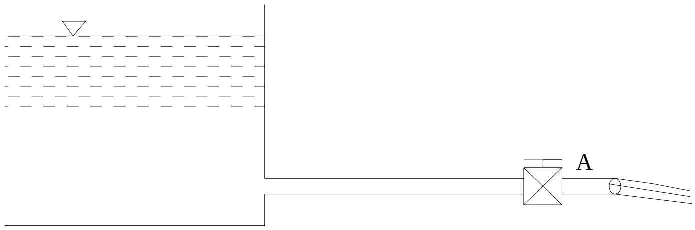


Fig. 3.1

There are, therefore, four possible types of flow.

1) *Steady uniform flow*. Conditions do not change with position or time. The velocity of fluid is the same at each cross-section; e.g. flow of a liquid through a pipe of constant diameter running completely full at constant velocity.
2) *Steady non-uniform flow*. Conditions change from point to point but not with time. The velocity and cross-sectional area of the stream may vary from cross-section to cross-section, they will not vary with time; e.g. flow of a liquid at a constant rate through a conical pipe running completely full.
3) *Unsteady uniform flow*. At a given instant of time the velocity at every point is the same, but this velocity will change with time; e.g. accelerating flow of a liquid through a pipe of uniform diameter running full, such as would occur when a pump is started up.
4) *Unsteady non-uniform flow*. The cross-sectional area and velocity vary from point to point and also change with time; a wave travelling along a channel.

3.3. STREAMLINES AND STREAM TUBES

If curves are drawn in a steady flow in such a way that the tangent at any point is in the direction of the velocity vector at that point, such curves are called *streamlines*. Individual fluid particles must travel on paths whose tangent is always in the direction of the fluid velocity at any point. Thus, path lines are the same as streamlines in steady flows.

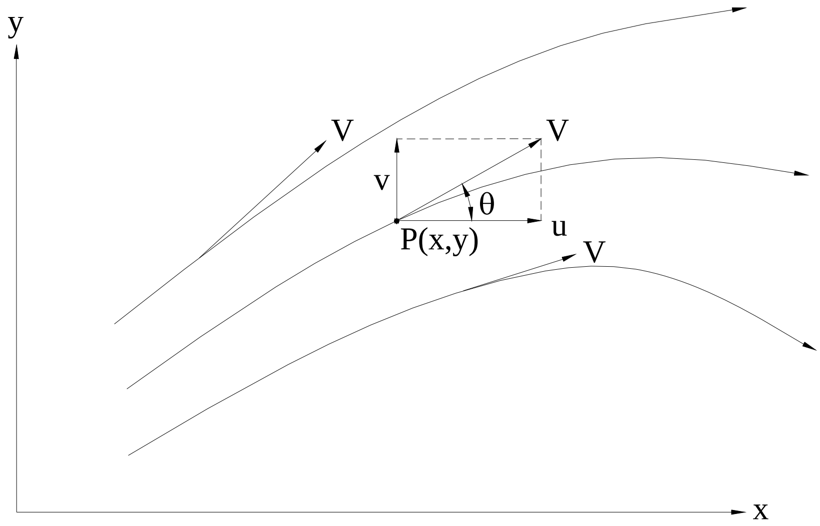


Fig. 3.2

Streamlines for a flow pattern in the xy-plane are shown in Fig. 3.2, in which a streamline passing through the point P (x, y) is tangential to the velocity vector $\vec{V}$ at P. If u and v are the x and y components of $\vec{V}$,

$$\frac{v}{u} = Tan\theta = \frac{dy}{dx}$$

Where dy and dx are the y and x components of the differential displacement ds along the streamline in the immediate vicinity of P. Therefore, the differential equation for streamlines in the xy-plane may be written as

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad udy - vdx = 0 \tag{3.6}$$

The differential equation for streamlines in space is,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{3.7}$$

Obviously, a streamline is everywhere tangent to the velocity vector; *there can be no flow occurring across a streamline*. In steady flow the pattern of streamlines remains invariant with time.

A *stream tube* such as that shown in Fig. 3.3 may be visualized as formed by a bundle of streamlines in a steady flow field. *No flow crosses the wall of a stream tube*. Often times in simpler flow problems, such as fluid flow in conduits, the solid boundaries may serve as the periphery of a stream tube since they satisfy the condition of having no flow crossing the wall of the tube.

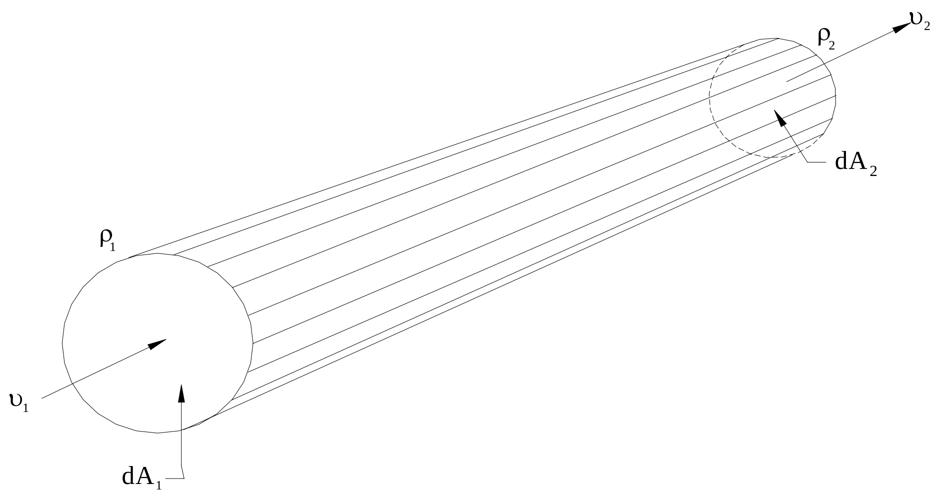


Fig. 3.3

In general, the cross-sectional area may vary along a stream tube since streamlines are generally curvilinear. Only in the steady flow field with uniform velocity will streamlines be straight and parallel. By definition, the velocities of all fluid particles in a uniform flow are the same in both magnitude and direction. If either the magnitude or direction of the velocity changes along any one streamline, the flow is then considered *non-uniform*.

3.4. ONE, TWO AND THREE-DIMENSIONAL FLOW

Although, in general, all fluid flow occurs in three dimensions, so that, velocity, pressure and other factors vary with reference to three orthogonal axes, in some problems the major changes occur in two directions or even in only one direction. Changes along the other axis or axes can, in such cases, be ignored without introducing major errors, thus simplifying the analysis.

Flow is described as *one-dimensional* if the factors, or parameters, such as velocity, pressure and elevation, describing the flow at a given instant, vary only along the direction of flow and not across the cross-section at any point. If the flow is unsteady, these parameters may vary with time. The one dimension is taken as the distance along the streamline of the flow, even though this may be a curve in space, and the values of velocity, pressure and elevation at each point along this streamline will be the average values across a section normal to the streamline (Fig.3.4).

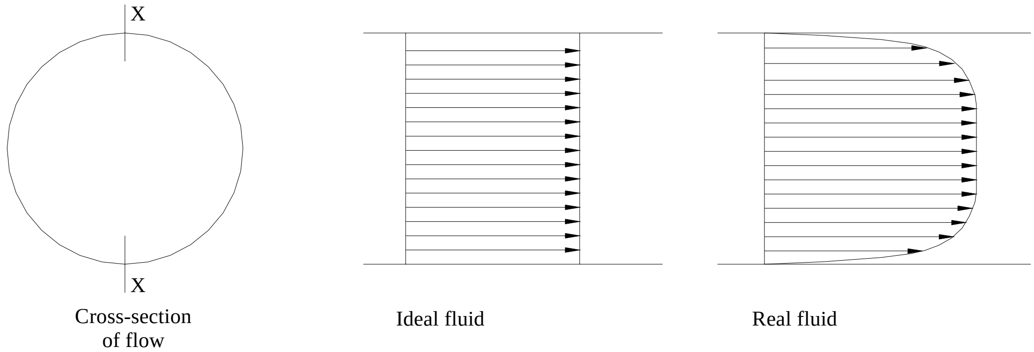


Fig. 3.4

In *two-dimensional* flow it is assumed that the flow parameters may vary in the direction of flow and in one direction at right angles, so that the streamlines are curves lying in a plane and identical in all planes parallel to this plane.

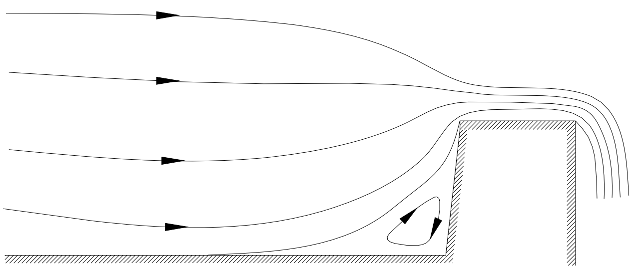


Fig. 3.5

Thus, the flow over a weir of constant cross-section (Fig.3.5) and infinite width perpendicular to the plane of the diagram can be treated as two-dimensional.

In *three-dimensional* flow it is assumed that the flow parameters may vary in space, x in the direction of motion, y and z in the plane of the cross-section.

3.5. EQUATION OF CONTINUITY: ONE-DIMENSIONAL STEADY FLOW

The application of the principle of conservation of mass to a steady flow in a stream tube results in the *equation of continuity*, which expresses the continuity of flow from section to section of the stream. Consider a physical system that is a particular collection of matter and is identified and viewed as being separated from everything external to the system by an imagined or real closed boundary. The fluid system retains its mass, but not its position or shape. This suggest needs to define a more convenient object for analysis. This objet is a volume fixed in space and is called a *control volume*, through whose boundary matter, mass, momentum, energy, and the like may flow. The boundary of the control volume is the *control surface*. The fixed control volume can be of any useful size (finite or infinitesimal) and shape, provided only that the bounding control surface is a closed (completely surrounding) boundary. Neither the control volume not the control shape changes shape or position with time.

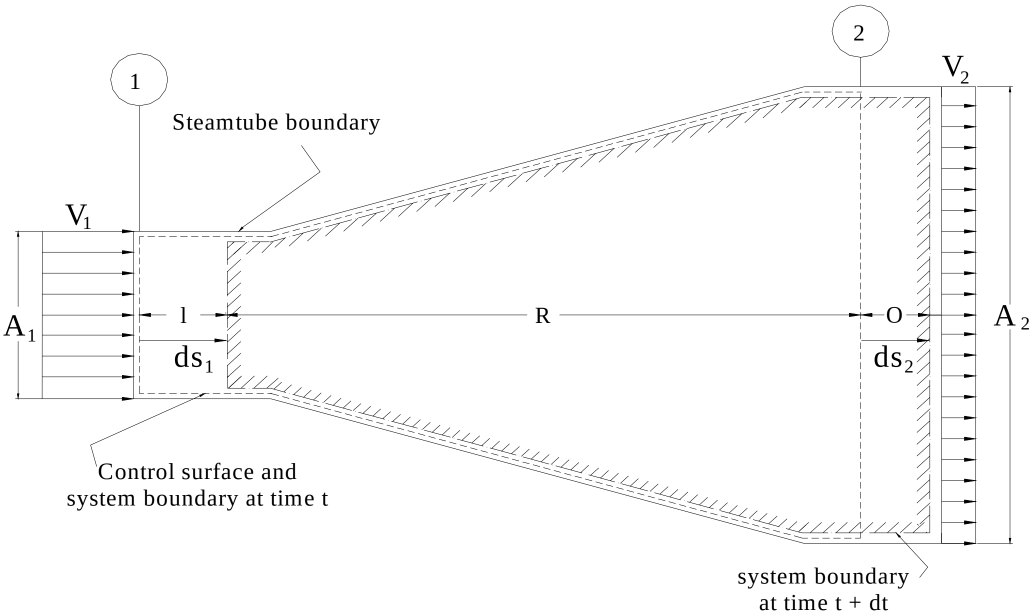


Fig. 3.6

Now consider the element of a finite stream tube in Fig. 3.6 through which passes a steady, one-dimensional flow of an incompressible fluid (note the uniform velocities at sections 1 and 2). In the tube near section 1 the cross-sectional area is A_1 and near section 2, A_2. With the control surface shown coinciding with the stream tube walls and the cross sections at 1 and 2, the control volume comprises volumes I and R. Let a fluid system be defined as the fluid within the control volume (I + R) at time t. The control volume is fixed in space, but in time dt the system moves downstream as shown. From the conservation of system mass

$$\left(m_I + m_R\right)_t = \left(m_R + m_O\right)_{t+dt}$$

(Mass of fluid in zones I and R at time t) = (Mass of fluid in zones O and R at time t+dt)

In a steady flow the fluid properties at points in space are not functions of time so $(m_R)_t = (m_R)_{t+dt}$ and consequently

$$(m_I)_t = (m_O)_{t+dt}$$

These two terms are easily in terms of the mass of fluid moving across the control surface in time dt. The volume of I is A_1ds_1, and that of O is A_2ds_2; accordingly,

$$(m_I)_t = \rho A_1 ds_1$$
$$(m_O)_{t+dt} = \rho A_2 ds_2$$

and

$$\rho A_1 ds_1 = \rho A_2 ds_2$$

Dividing by dt,

$$\rho A_1 \frac{ds_1}{dt} = \rho A_2 \frac{ds_2}{dt}$$

However, ds_1/dt and ds_2/dt are recognized as the velocities past sections 1 and 2, respectively, therefore, if m = ρAV is the mass flow rate, then

$$m = \rho A_1 V_1 = \rho A_2 V_2$$

$$Q = A_1 V_1 = A_2 V_2 \qquad (3.8)$$

Which is the *equation of continuity*. Thus for incompressible fluids, along a stream tube the product of velocity and cross-sectional area will be constant. This product, Q, s designated as *(flowrate) discharge* and has dimensions of $[L^3T^{-1}]$ and units of cubic meters per second (m^3/sec).

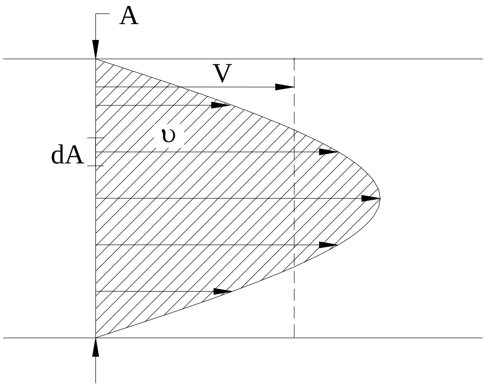


Fig. 3.7

Frequently in fluid flows the velocity distribution through a flow cross-section may be non-uniform, as shown in Fig. 3.7. From consideration of mass, it is evident at once that non-uniformity of velocity distribution does not invalidate the continuity principle. Thus, for steady flow of the incompressible fluid, Equ. (3.8) applies as before. Here, however, the velocity V in the equation is the *mean velocity* defined by V = Q/A in which the discharge Q is obtained from the summation of the differential discharges, dQ, passing through the differential areas, dA. Thus, V is a fictitious uniform velocity that will transport the same amount of mass through the cross-section as will the actual the velocity distribution;

$$V = \frac{1}{A}\int_A v\,dA \qquad (3.9)$$

From which the mean velocity may be obtained by performing the indicated integration (Equ. 3.9). With the velocity mathematically defined, formal integration may be employed; when the velocity profile is known but not mathematically defined, graphical or numerical methods may be used to evaluate integral.

The fact that the product AV remains constant along a stream tube allows a partial physical interpretation of streamline pictures. As the cross-sectional area of a stream tube increases, the velocity must decrease; hence the conclusion; streamlines widely spaced indicate regions of low velocity, streamlines closely spaced indicate regions of high velocity.

The continuity of flow equation is one of he major tools of fluid mechanics, providing a means of calculating velocities at different points in a system.

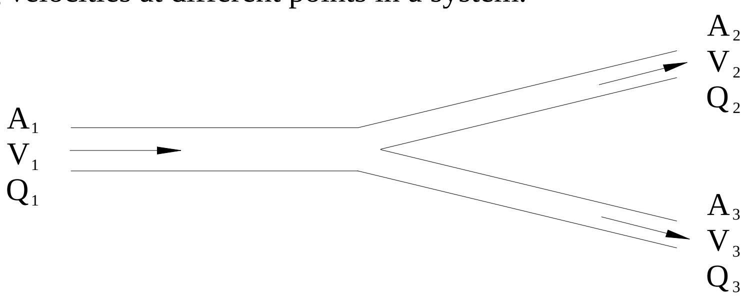


Fig. 3.8

The continuity equation can also be applied to determine the relation between the flows into and out of a junction. In Fig. 3.8, for steady conditions,

Total inflow to junction = Total outflow from junction

$$Q_1 = Q_2 + Q_3$$

or

$$A_1V_1 = A_2V_2 + A_3V_3$$

In general, if we consider flow towards the junction as positive and flow away from the junction as negative, then for steady flow at any junction the algebraic sum of the discharges must be zero.

$$\sum Q = 0$$

EXAMPLE 3.1: Water flows through a pipe AB (Fig.3.9) of diameter d_1 = 50 mm, which is in series with a pipe BC of diameter d_2 = 75 mm in which the mean velocity V_2 = 2 m/sec.

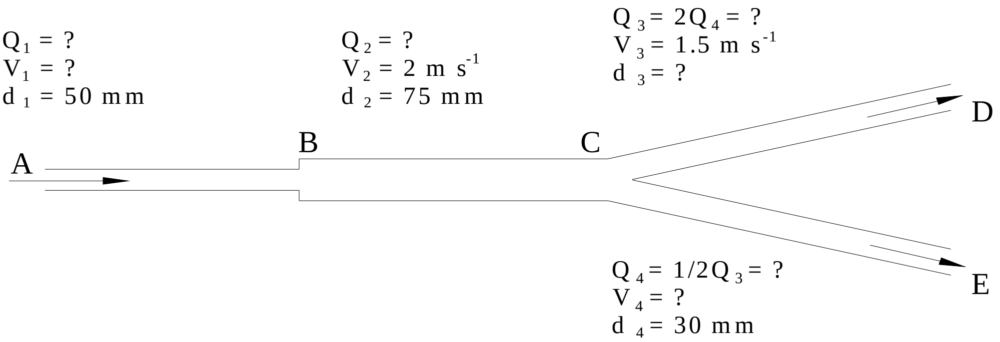


Fig. 3.9

At C the pipe forks and one branch CD is of diameter d_3 such that the mean velocity $V_3 = 1.5$ m/sec. The other branch CE is of diameter $d_4 = 30$ mm and conditions are such that the discharge Q_2 from BC divides so that $Q_4 = 0.5Q_3$. Calculate the values of Q_1, V_1, Q_2, Q_3, d_3, Q_4 and V_4.

SOLUTION: Since pipes AB and BC in series, the volume rate of flow (discharge) will be the same in each pipe, $Q_1 = Q_2$.

Q_2 = Area of pipe×Mean velocity $= \dfrac{\pi d_2^2}{4} V_2$

$$Q_1 = Q_2 = \frac{\pi}{4} \times 0.075^2 \times 2 = 8.836 \times 10^{-3}\ m^3/\text{sec}$$

Mean velocity in AB = $V_1 = \dfrac{4Q_1}{\pi d_1^2} = \dfrac{4 \times 8.836 \times 10^{-3}}{\pi \times 0.05^2}$

$$V_1 = 4.5\,m/\text{sec}$$

Considering pipes BC, CD and DE, the discharge from BC must be equal to the sum of discharges through CD and CE. Therefore, $Q_2 = Q_3 + Q_4$, and since $Q_4 = 0.5Q_3$, we have $Q_2 = 1.5Q_3$, from which

$$Q_3 = \frac{Q_2}{1.5} = \frac{8.836 \times 10^{-3}}{1.5} = 5.891 \times 10^{-3}\ m^3/\text{sec}$$

and

$$Q_4 = \frac{Q_3}{2} = 2.945 \times 10^{-3}\ m^3/\text{sec}$$

Also, since

$$Q_3 = \frac{\pi d_3^2}{4} V_3 \quad , \; d_3 = \sqrt{\frac{4Q_3}{\pi V_3}}$$

$$d_3 = \sqrt{\frac{4 \times 5.891 \times 10^{-3}}{\pi \times 1.5}} = 0.071m$$

$$V_4 = \frac{4Q_4}{\pi d_4^2} = \frac{4 \times 2.945 \times 10^{-3}}{\pi \times 0.03^2} = 4.17\, m/\sec$$

CHAPTER 4

BASIC EQUATIONS FOR ONE-DIMENSIONAL FLOW

4.1. EULER'S EQUATION OF MOTION

Consider a streamline and select a small cylindrical fluid system for analysis as shown in Fig. 4.1.

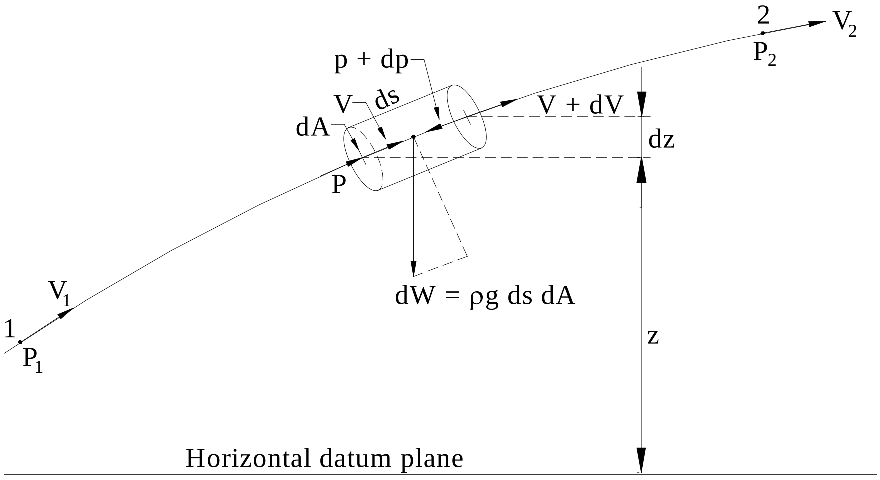


Fig. 4.1

The forces tending to accelerate the cylindrical fluid system are: forces on the ends of the system,

$$pdA - (p + dp)dA = -dpdA$$

and the component of weight in the direction of motion,

$$-\rho g dsdA\frac{dz}{ds} = -\rho gdAdz$$

The differential mass being accelerated by the action of these differential forces is,

$$dm = \rho dsdA$$

Applying Newton's second law dF = dm×a along the streamline and using the one-dimensional expression for acceleration gives

$$-dpdA - \rho gdAdz = (\rho dsdA)V\frac{dV}{ds}$$

Dividing by ρdA produces the one dimensional Euler equation,

$$\frac{dp}{\rho} + VdV + gdz = 0$$

This equation is divided by g and written

$$d\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right) = 0$$

4.2. BERNOULLI'S EQUATION

The one-dimensional Euler equation can be easily integrated between any points (because γ and g are both constants) to obtain

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

As points 1 and 2 are any two arbitrary points on the streamline, the quantity

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = H = \text{Constant} \qquad (4.1)$$

Applies to all points on the streamline and thus provides a useful relationship between pressure p, the magnitude V of the velocity, and the height z above datum. Equ. (4.1) is known as the *Bernoulli equation* and the *Bernoulli constant H* is also termed the *total head*.

Examination of the Bernoulli terms of Equ. (4.1) reveals that p/γ and z are respectively, the pressure (either gage or absolute) and potential heads and may be visualized as vertical distances. The sum of velocity head $V^2/2g$ and pressure head p/γ could be measured by placing a tiny open tube in the flow with its open end upstream. Thus Bernoulli equation may be visualized for liquids as in Fig. 4.2, the sum of the terms (total head) being the constant distance between the horizontal datum plane and the *total headline* or *energy line (E.L.)*. The *piezometric head line* or *hydraulic grade line (H.G.L.)* drawn through the tops of the piezometer columns gives a picture of the pressure variation in the flow; evidently

1) Its distance from the stream tube is a direct measure of the static pressure in the flow,
2) Its distance below the energy line is proportional to the square of the velocity.

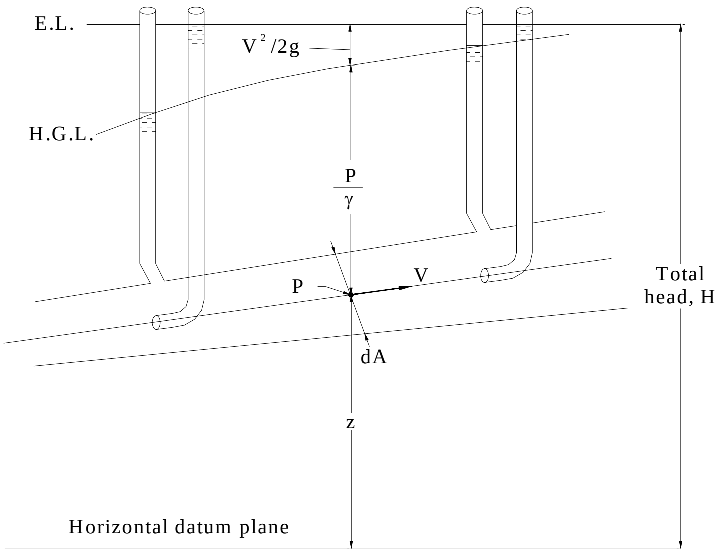


Fig. 4.2

4.3. MECHANICAL ENERGY OF A FLOWING FLUID

An element of fluid, as shown in Fig. 4.3, will posses potential energy due to its height z above datum and kinetic energy due to its velocity V, in the same way as any other object.

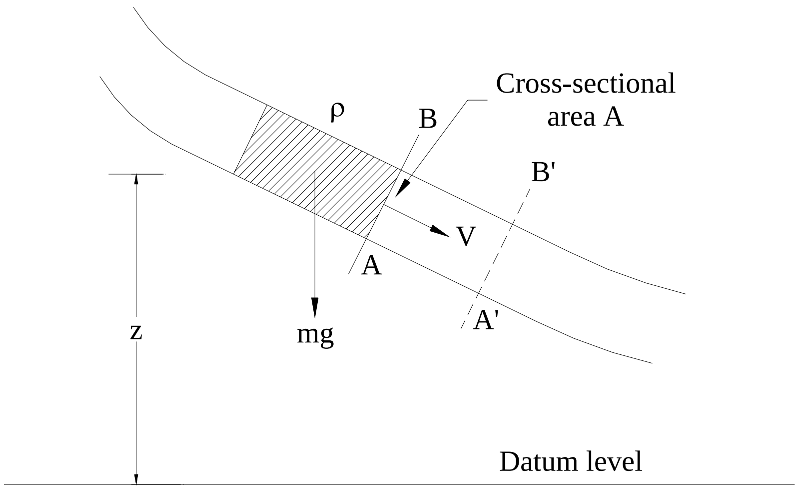


Fig. 4.3

For an element of weight mg,

Potential energy = mgz

Potential energy per unit weight = z (4.2)

Kinetic energy = $\frac{1}{2}mV^2$

Kinetic energy per unit weight = $\frac{V^2}{2g}$ (4.3)

A steadily flowing stream of fluid can also do work because of its pressure. At any given cross-section, the pressure generates a force and, as the fluid flows, this cross-section will move forward and so work will be done. If the pressure at section AB is p and the area of the cross-section is A,

Force exerted on AB = pA

After a weight mg of fluid has flowed along the stream tube, section AB will have moved to A'B':

Volume passing AB = $\frac{mg}{\rho g} = \frac{m}{\rho}$

Therefore,

Distance AA' = $\frac{m}{\rho A}$

Work done = Force×Distance AA'

$$= pA \times \frac{m}{\rho A}$$

Work done per unit weight = $\frac{p}{\rho g} = \frac{p}{\gamma}$ (4.4)

The term p/γ is known as the flow work or *pressure energy*. Note that term pressure energy refers to the energy of a fluid when flowing under pressure. The concept of pressure energy is sometimes found difficult to understand. In solid body mechanics, a body is free to change its velocity without restriction and potential energy can be freely converted to kinetic energy as its level falls. The velocity of a stream of fluid which has a steady volume rate of flow (discharge) depends on the cross-sectional area of the stream. Thus, if the fluid flows in a uniform pipe, its velocity cannot change and so the conversion of potential energy to kinetic energy cannot take place as the fluid loses elevation. The surplus energy appears in the form of an increase in pressure. As a result, pressure energy can be regarded as potential energy in transit.

Comparing the results obtained in Eqs. (4.2), (4.3) and (4.4) with Equ. (4.1), it can be seen that the three terms Bernoulli's equation are the pressure energy per unit weight, the kinetic energy per unit weight, and the potential energy per unit weight; the constant H is the total energy per unit weight. Thus, Bernoulli's equation states that, for steady flow of a frictionless fluid along a streamline, the total energy per unit weight remains constant from point to point although its division between the three forms of energy may vary:

$$\begin{matrix}\text{Pressure} & & \text{Kinetic} & & \text{Potential} & & \text{Total} & \\ \text{energy per} & + & \text{energy per} & + & \text{energy per} & = & \text{energy per} & = \text{Constant} \\ \text{unit weight} & & \text{unit weight} & & \text{unit weight} & & \text{unit weight} & \end{matrix}$$

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = H \qquad (4.5)$$

Each of these terms has the dimensions of a length, or head, and they are often referred to as the *pressure head p/γ*, the *velocity head* $V^2/2g$, the *potential head z* and the *total head H*. Between any two points, suffixes 1 and 2, on a streamline, Equ. (4.5) gives

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2 \qquad (4.6)$$

or

Total energy per unit weight at 1 = Total energy per unit weight at 2

In formulating Equ. (4.6), it has been assumed that no energy has been supplied to or taken from the fluid between points 1 and 2. Energy could have been supplied by introducing a pump; equally, energy could have been lost by doing work against friction in a machine such as a turbine. Bernoulli's equation can be expanded to include these conditions, giving

Total energy per unit weight at 1	=	Total energy per unit weight at 2	+	Loss Per unit weight	+	Work done per unit weight	-	Energy supplied per unit weight

EXAMPLE 4.1: A fire engine develops a head of 50 m, i.e., it increases the energy per unit weight of the water passing through it by 50 m. The pump draws water from a sump at A through a 150 mm diameter pipe in which there is a loss of energy per unit weight due to friction $h_1 = 5V_1^2/2g$ varying with the mean velocity V_1 in the pipe, and discharges it through a 75 mm nozzle at C, 30 m above the pump, at the end of a 100 mm diameter delivery pipe in which there is a loss of energy per unit weight $h_2 = 12V_2^2/2g$. Calculate,

a) The velocity of the jet issuing from the nozzle at C,
b) The pressure in the suction pipe at the inlet to the pump at B.

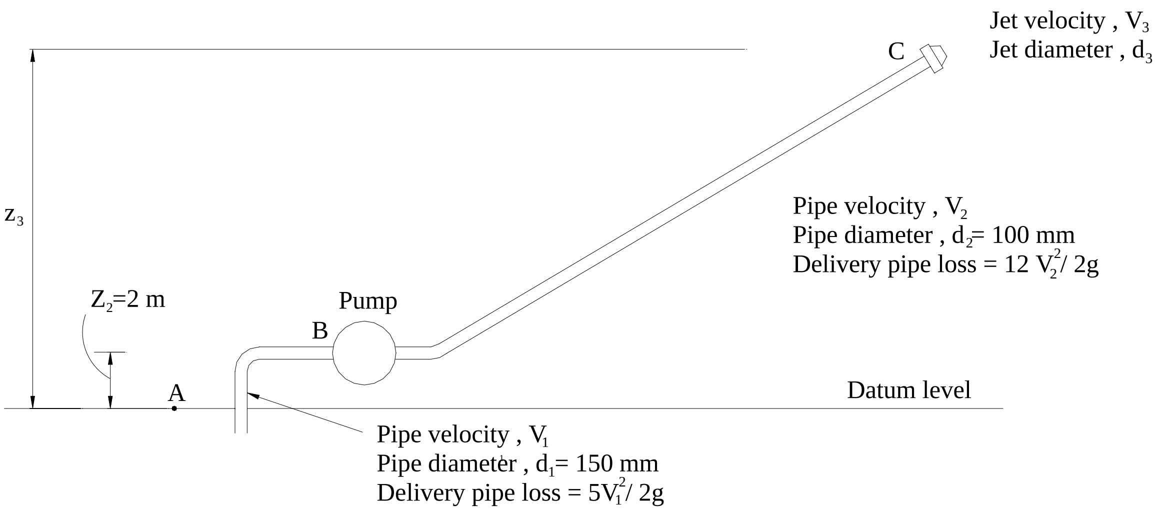


Fig.4.4

SOLUTION:

a) We can apply Bernoulli's equation in the form of Equ. (4.6) between two points, one of which will be C, since we wish to determine the jet velocity V_3, and the other point at which conditions are known, such as a point A on the free surface of the sump where the pressure will be atmospheric, so that $p_A = 0$, the velocity V_A will be zero if the sump is large, and A can be taken as datum level so that $z_A = 0$. Then,

Total energy per unit weight at A	=	Total energy per unit weight at C	+	Loss in inlet pipe	-	Energy per unit weight supplied by pump	+	Loss in discharge pipe	(I)

Total energy per unit weight at A $= \dfrac{p_A}{\gamma} + \dfrac{V_A^2}{2g} + z_A = 0$

Total energy per unit weight at C $= \dfrac{p_C}{\gamma} + \dfrac{V_3^2}{2g} + z_3$

p_C = Atmospheric pressure = 0

$z_3 = 30 + 2 = 32m$

Therefore,

Total energy per unit weight at C $= 0 + \dfrac{V_3^2}{2g} + 32 = \dfrac{V_3^2}{2g} + 32$

Loss in inlet pipe, $h_1 = 5\dfrac{V_1^2}{2g}$

Energy per unit weight supplied by pump = 50 m

Loss in delivery pipe, $h_2 = 12\dfrac{V_2^2}{2g}$

Substituting in (I),

$$0 = \frac{V_3^2}{2g} + 32 + 5\frac{V_1^2}{2g} - 50 + 12\frac{V_2^2}{2g}$$

$$V_3^2 + 5V_1^2 + 12V_2^2 = 2 \times g \times 18 \qquad \text{(II)}$$

From the continuity of flow equation,

$$\frac{\pi d_1^2}{4}V_1 = \frac{\pi d_2^2}{4}V_2 = \frac{\pi d_3^2}{4}V_3$$

Therefore,

$$V_1 = \left(\frac{d_3}{d_1}\right)^2 V_3 = \left(\frac{75}{150}\right)^2 V_3 = \frac{1}{4}V_3$$

$$V_2 = \left(\frac{d_3}{d_2}\right)^2 V_3 = \left(\frac{75}{100}\right)^2 V_3 = \frac{9}{16}V_3$$

Substituting in Equ. (II),

$$V_3^2\left[1+5\left(\frac{1}{4}\right)^2+12\left(\frac{9}{16}\right)^2\right] = 2\times g\times 1$$

$$5.109V_3^2 = 2\times g\times 18$$

$$V_3 = 8.31\,m/\sec$$

b) If p_B is the pressure in the suction pipe at the pump inlet, applying Bernoulli's equation to A and B,

Total energy per unit weight at A	=	Total energy per unit weight at B	+	Loss in inlet pipe

$$0 = \frac{p_B}{\gamma} + \frac{V_1^2}{2g} + z_2 + 5\frac{V_1^2}{2g}$$

$$\frac{p_B}{\gamma} = -z_2 - 6\frac{V_1^2}{2g}$$

$$z_2 = 2m \quad , V_1 = \frac{1}{4}V_3 = \frac{8.31}{4} = 2.08\,m/\sec$$

$$\frac{p_B}{\gamma} = -2 - 6\frac{2.08^2}{2g} = -3.32m$$

$$p_B = -3.32\times 1 = -3.32\,t/m^2 \quad \text{(below atmospheric pressure)}$$

4.4. THE WORK-ENERGY EQUATION

The application of work-energy principles to fluid results in a powerful relationship between fluid properties, work done, and energy transported. The Bernoulli equation is then seen to be equivalent to the mechanical work-energy equation for ideal fluid flow.

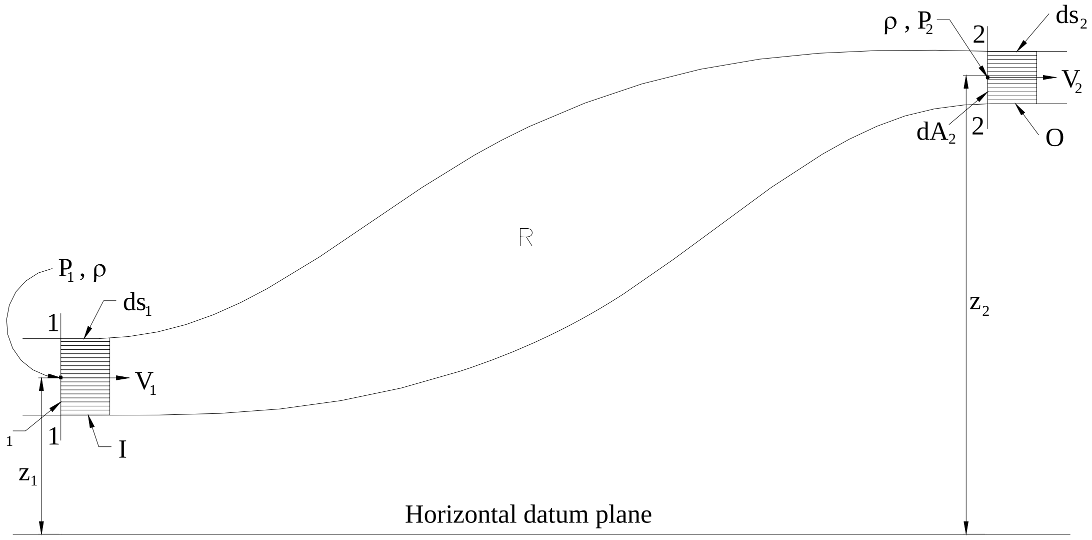


Fig. 4.5

Consider the differential stream tube section shown in Fig. 4.5 and the fluid system that occupies zones I and R of the control volume 1221 at time t and zones R and O at time t+dt. For steady flow the continuity Equ. (3.8) gives

$$dA_1V_1 = dA_2V_2 \qquad \text{or} \qquad dA_1ds_1 = dA_2ds_2$$

From dynamics, the mechanical work-energy relation (which is only an integrated form of Newton's second law) states that the work dW (expressed as a force acting over a distance) done on a system produces an equivalent change in the sum of the kinetic (KE) and potential (PE) energies of the system, that is, in time dt

$$dW = d(KE+PE) = (KE+PE)_{t+dt} - (KE+PE)_t$$

Now

$$(KE+PE)_t = (KE+PE)_R + (KE+PE)_I$$

$$(KE+PE)_t = (KE+PE)_R + \frac{1}{2}(\rho dA_1ds_1)V_1^2 + \gamma(dA_1ds_1)z_1$$

$$(KE+PE)_{t+dt} = (KE+PE)_R + (KE+PE)_O$$

$$(KE+PE)_{t+dt} = (KE+PE)_R + \frac{1}{2}(\rho dA_2ds_2)V_2^2 + \gamma(dA_2ds_2)z_2$$

Because kinetic energy of translation is $mV^2/2$ and potential energy is equivalent to the work of raising the weight of fluid in a zone to a height z above the datum.

The external work done on the system is all accomplished on cross sections 11 and 22 because there is no motion perpendicular to the stream tube so the lateral pressure forces can do no work. Also because all internal forces appear in equal and opposite pairs, there is no network done internally. The work done by the fluid entering I on the system in time dt is the *flow work.*

$$(p_1 dA_1)ds_1$$

As the system does work on the fluid in time O in time dt, the work done on the system is

$$-(p_2 dA_2)ds_2$$

In sum then

$$(p_1 dA_1)ds_1 - (p_2 dA_2)ds_2 = \frac{1}{2}(\rho dA_2 ds_2)V_2^2 + \gamma(dA_2 ds_2)z_2 - \frac{1}{2}(\rho dA_1 ds_1)V_1^2 - \gamma(dA_1 ds_1)z_1$$

Dividing by $dA_1 ds_1 = dA_2 ds_2$ produces

$$p_1 - p_2 = \frac{1}{2}\rho V_2^2 + \gamma z_2 - \frac{1}{2}\rho V_1^2 - \gamma z_1$$

When rearranged, this is recognized as Bernoulli’s equation,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \qquad (4.7)$$

Which can be interpreted now as a mechanical energy equation. Terms such as p_1/γ, $V_1^2/2g$, z have the units of meters which represent energy per unit weight of fluid.

4.5. KINETIC ENERGY CORRECTION FACTOR

The derivation of Bernoulli’s equation has been carried out for a stream tube assuming a uniform velocity across the inlet and outlet sections. In a real fluid flowing in a pipe or over a solid surface, the velocity will be zero at the solid boundary and will increase as the distance from the boundary increases. The kinetic energy per unit weight of the fluid will increase in a similar manner. If the cross-section of the flow is assumed to be composed of a series of small elements of area dA and the velocity normal to each element is u, the total kinetic energy passing through the whole cross-section can be found by determining the kinetic energy passing through an element in unit time and then summing by integrating over the whole area of the section,

$$\text{Kinetic energy } = \int_A \frac{u^2}{2} dm = \int_A \frac{u^2}{2} \rho u dt dA = \frac{\rho dt}{2}\int_A u^3 dA$$

Which can be readily integrated if the exact velocity is known. It is, however, more convenient to express the kinetic energy in terms of average velocity V at the section and a *kinetic energy correction factor* α such that

$$\text{K.E.} = \alpha \frac{V^2}{2} m = \alpha \frac{\rho dt}{2} A V^3$$

In which m = ρAVdt is the total mass of the fluid flowing across the cross-section during dt. By comparing the two expressions for kinetic energy, it is obvious that

$$\alpha = \frac{1}{AV^3} \int_A u^3 dA \qquad (4.8)$$

Mathematically, the cube of the average is less than the average of cubes, that is,

$$V^3 < \frac{1}{A} \int_A u^3 dA$$

The numerical value of α will always be greater than 1. Then Equ. (4.6) takes the form of Equ. (4.9) by taking kinetic energy correction factor, α, as

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 \qquad (4.9)$$

The factor α depends on the shape of the cross-section and the velocity distribution. In most engineering problems of turbulent flow in circular pipes, α has a numerical value ranging from 1.01 to 1.10. Here α can be assumed to be unity without introducing any serious error.

The energy equation (Equ.4.9) for the flow system becomes

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \qquad (4.7)$$

Which is identical to the energy equation for fluid flow along a streamline.

4.6. APPLICATIONS OF BERNOULLI'S EQUATION

Although there is always some friction loss in the flow of real fluids, in many engineering problems the assumption of frictionless flow may yield satisfactory results.

An important feature of Bernoulli's equation is in its graphical representation of the three terms, p/γ, z, and $V^2/2g$, at each section of the flow system. In Fig. 4.6 there is shown a typical example of steady flow of an *ideal* fluid from a large reservoir through a system of pipes varying in size and terminating in a nozzle.

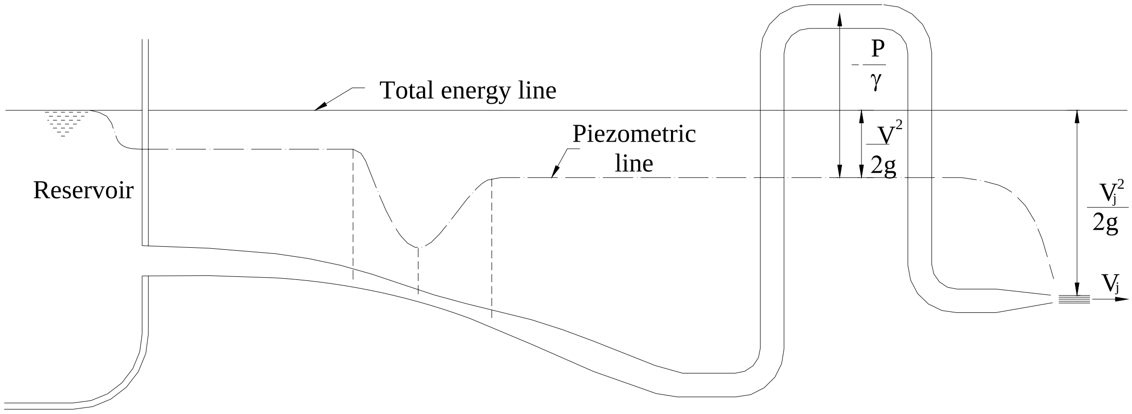


Fig. 4.6

Since the cross-sectional area of the reservoir is very large and there the velocity is zero, both the total energy line and piezometric line coincide with the free surface in the reservoir. The total energy line of the whole flow system must be horizontal if the flow is assumed to be frictionless (ideal). The flow in the pipes at various sections follows the continuity principle, that is, Q = VA. The volume rate of flow (discharge) Q can be determined by writing Bernoulli's equation to relate flow conditions at the free surface of the reservoir and the jet at the nozzle outlet. Thus the velocity head $V^2/2g$ at any section is determined and, finally the piezometric line is sketched in at a distance of $V^2/2g$ below the total energy line. The distance between the piezometric line and the line of the pipe is the pressure head p/γ at any section. If the piezometric line stays above the centerline of the pipe, the pressure head is positive. Otherwise, negative pressure prevails throughout the region where the piezometric line falls below the centerline of the pipe. Care must be taken in dealing with the regions of negative pressure because of the adverse effect of *cavitation*, a phenomenon closely associated with the regions where the local pressure drops to the *vapor pressure* of the liquid flowing in the system. At the region of low pressure liquid tends to vaporize and water bubbles start to form. These vapor bubbles are carried downstream and subsequently collapse in the zones of higher pressure. The repeated collapsing of vapor bubbles produces extremely high hydrodynamic pressures upon the solid boundaries, frequently causing severe physical damages to the boundary material. In order to avoid cavitation, it is essential to eliminate the zone of low pressure where vaporization may take place. The position of the piezometric line yields visual information on the pressure conditions of the whole flow system.

EXAMPLE 4.2: The 10 cm (diameter) siphon shown in Fig. 4.7 is filled with water and discharging freely into the atmosphere through a 5 cm (diameter) nozzle. Assume frictionless flow and find, a) the discharge in cubic meter per second and, b) the pressure at C, D, E, and F.

SOLUTION:

a) Since the flow is assumed to be frictionless, Bernoulli's equation (Equ. 4.6) can be applied to points A and G with elevation datum at G and zero gage pressure as pressure datum:

$$\frac{p_A}{\gamma}+z_A+\frac{V_A^2}{2g}=\frac{p_G}{\gamma}+z_G+\frac{V_G^2}{2g}$$

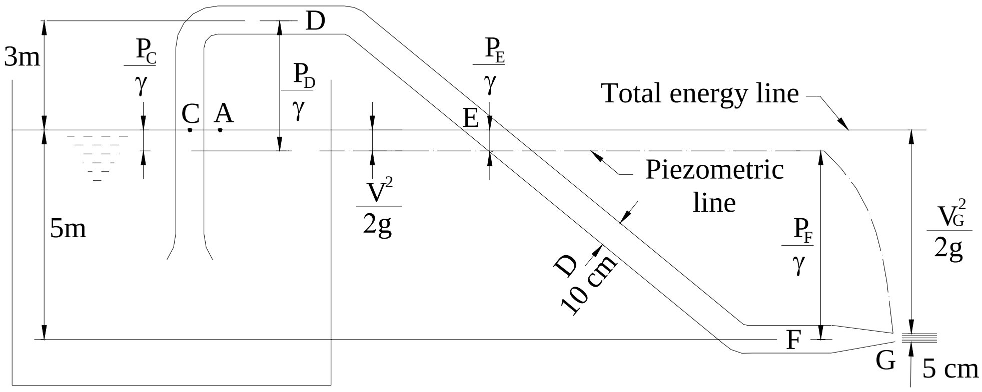


Fig. 4.7

Point A is at the reservoir surface where atmospheric pressure prevails and velocity is negligible. Therefore,

$$0+5+0=0+0+\frac{V_G^2}{2g}$$

$$V_G=\sqrt{2g\times 5}=9.90\,m/\sec$$

$$Q=A_G V_G=\frac{\pi}{4}\times 0.05^2\times 9.90=0.0194\,m^3/\sec$$

b) From the continuity equation the velocity in the 10 cm diameter siphon is,

$$V=\frac{Q}{A}=\frac{4Q}{\pi D^2}=\frac{4\times 0.0194}{\pi\times 0.10^2}=2.48\,m/\sec$$

By using the same elevation and pressure data as in (a) and writing Bernoulli's equation to relate flow conditions at A and C,

$$0+5+0=\frac{p_C}{\gamma}+5+\frac{2.48^2}{2g}$$

from which

$$\frac{p_C}{\gamma}=-0.31m \text{ of water}$$

$$p_C=-0.31t/m^2$$

The minus sign indicates vacuum.

Similarly, between A and D

$$0+5+0=\frac{p_D}{\gamma}+8+\frac{2.48^2}{2g}$$

$$\frac{p_D}{\gamma}=-3.31m \text{ of water}$$

$$p_D=-3.31t/m^2$$

Between A and E,

$$0+5+0=\frac{p_E}{\gamma}+5+\frac{2.48^2}{2g}$$

$$\frac{p_E}{\gamma}=-0.31m \text{ of water}$$
$$p_E=-0.31t/m^2$$

Between A and F,

$$0+5+0=\frac{p_F}{\gamma}+0+\frac{2.48^2}{2g}$$

$$\frac{p_F}{\gamma}=4.69m \text{ of water}$$
$$p_F=4.69t/m^2$$

The total energy line and piezometric line shown in Fig. 4.7 are self-explanatory.

4.6.1. Torricelli's Theorem

The classical *Torricelli's theorem*, which was formulated through experimentation, states that the velocity of liquid flowing out of an *orifice* is proportional to the square root of the height of liquid above the center of the orifice. This statement can readily be proved by applying Bernoulli's equation.

The reservoir in Fig. 4.8 is filled with a liquid of specific weight γ to a height h above the center of a round orifice O in its side. It is assumed, 1) That both the free surface of the liquid in the reservoir and the liquid jet are exposed to the atmospheric pressure, that is, $p_A = p_0$; 2) That the liquid surface in the reservoir remains constant; and 3) That the surface area of the reservoir is large compared with the cross-sectional area of the orifice. Thus the velocity head at A may be neglected, that is, $V_A^2/2g \cong 0$. Therefore, if the elevation at the center of the orifice is taken as a datum, Bernoulli's equation for flow conditions between A and O becomes,

$$\frac{p_A}{\gamma}+h+0=\frac{p_O}{\gamma}+0+\frac{V_o^2}{2g}$$

or, solving for V_o,

$$V_O=\sqrt{2gh} \qquad (4.10)$$

Which is known as Torricelli's theorem. It is worth noting here that the *velocity of efflux* is identical to the velocity attained in free fall.

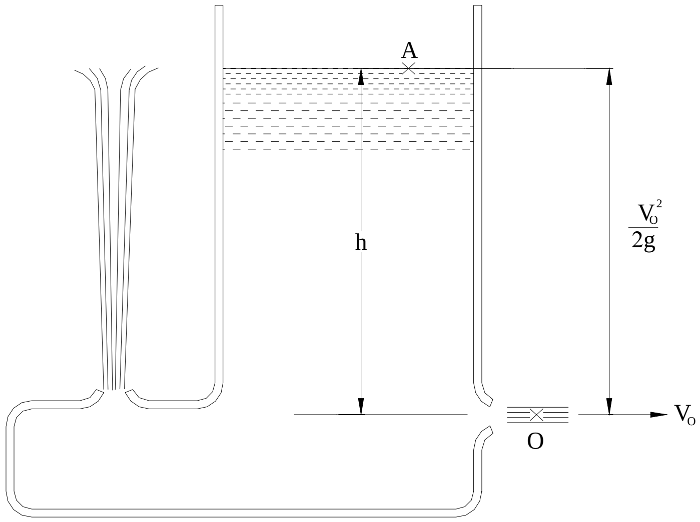


Fig. 4.8

EXAMPLE 4.3: The pressurized tank shown has a circular cross-section 2 m in diameter. Oil is drained through a nozzle 0.08m in diameter in the side of the tank. Assuming that the air pressure is maintained constant, how long does it take to lower the oil surface in the tank by 2 m? The specific weight of the oil in the tank is 0.75 t/m^3 and that of mercury is 13.6 t/m^3.

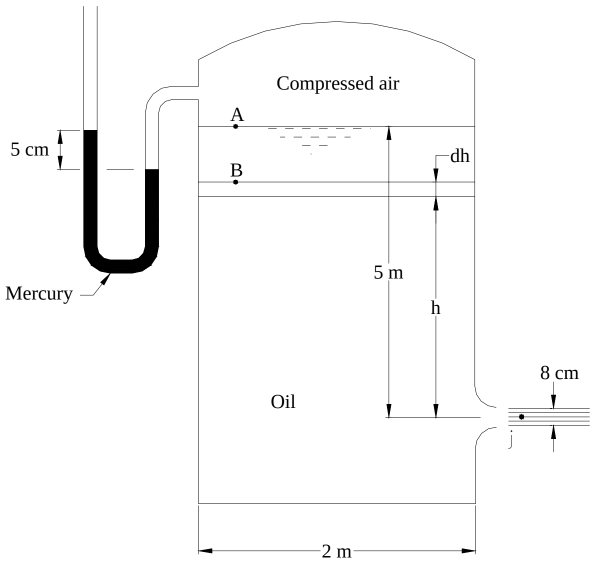


Fig. 4.9

SOLUTION: The pressure head in the tank above the oil surface is maintained at a constant value of,

$$\frac{p_A}{\gamma_{oil}} = 0.05 \times \frac{13.6}{0.75} = 0.91m \text{ of oil}$$

Since the oil surface drops constantly, the discharge out of the nozzle will vary with time. By neglecting the friction loss, Bernoulli's equation (Equ. 4.6) is written for flow conditions between a point B in the oil when the oil surface is at a height h above the center of the nozzle and the point j in the jet.

$$\frac{p_B}{\gamma_{oil}} + z_B + \frac{V_B^2}{2g} = \frac{p_j}{\gamma_{oil}} + z_j + \frac{V_j^2}{2g}$$

The cross-sectional area of the tank is very much larger than the jet area; V_B is approximately zero for all practical purposes. The jet issuing out of the nozzle is subject to the atmospheric pressure. By choosing the center of the nozzle as the elevation datum and the local atmospheric pressure as the pressure datum,

$$0.91 + h + 0 = 0 + 0 + \frac{V_j^2}{2g}$$

$$V_j = \sqrt{2g(0.91 + h)}$$

Which is the instantaneous velocity of the jet when the oil surface is at h above the center of the nozzle.

From the continuity equation the total discharge out of the nozzle during a small interval dt must be equal to the reduction in the volume of oil in the tank. Thus,

$$A_j V_j dt = A_t dh$$

or

$$\frac{\pi}{4} \times 0.08^2 \sqrt{2g(h + 0.91)}dt = -\frac{\pi}{4} \times 2^2 \times dh$$

from which

$$\int_0^t dt = -\frac{625}{\sqrt{2g}} \int_0^3 (0.91 + h)^{-1/2} dh$$

By integrating

$$t = -\frac{625}{\sqrt{2g}} \left[2(0.91 + h)^{1/2}\right]_5^3 = 128 \sec$$

4.6.2. Free Liquid Jet

A free liquid jet is actually a streamline along which the pressure is atmospheric. Accordingly, Bernoulli's equation may be applied to the whole *trajectory* of the jet if the air resistance is neglected. With the pressure head p/γ equal to zero, the sum of velocity head and elevation head above any arbitrarily chosen datum must remain constant for all points along the trajectory (Fig. 4.10).

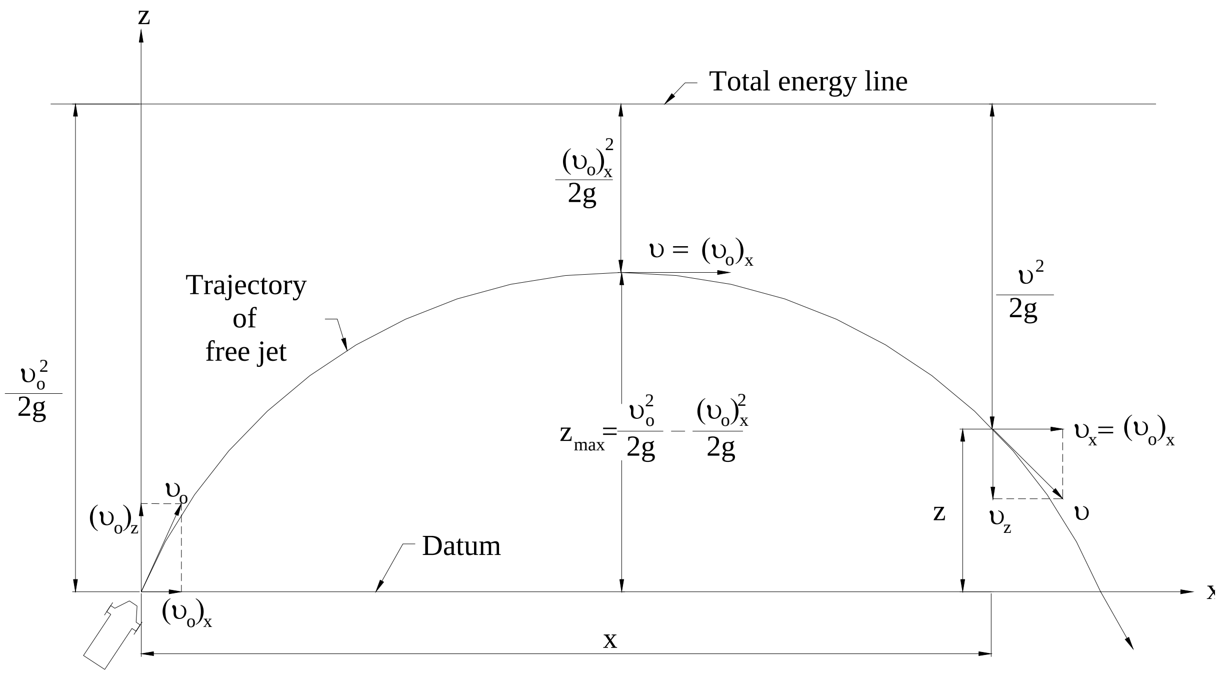


Fig. 4.10

Therefore,

$$z + \frac{V^2}{2g} = \text{Constant}$$

The total energy line is shown to be horizontal and at a distance of $V^2/2g$ above trajectory. The velocity at any point of the jet may be determined from its components V_x and V_y. Thus V equals $(V_x^2 + V_y^2)^{1/2}$. Here basic equations of projectile motion in physics are used to determine the velocity components at any point along the trajectory:

$$V_x = (V_0)_x$$
$$V_z = (V_o)_z - gt$$

The coordinates of the trajectory are expressed as follows:

$$x = (V_0)_x t$$
$$z = (V_0)_z t - \frac{1}{2} g t^2$$

Where t is the elapsed after the liquid jet leaves the nozzle.

EXAMPLE 4.4: Water is discharged from a 5 cm (diameter) nozzle, which is inclined at a 30^0 angle above the horizontal. If the jet strikes the ground at a horizontal distance of 5 m and a vertical distance of 1 m from the nozzle as shown in Fig. 4.11, what is the discharge in cubic meter per second?

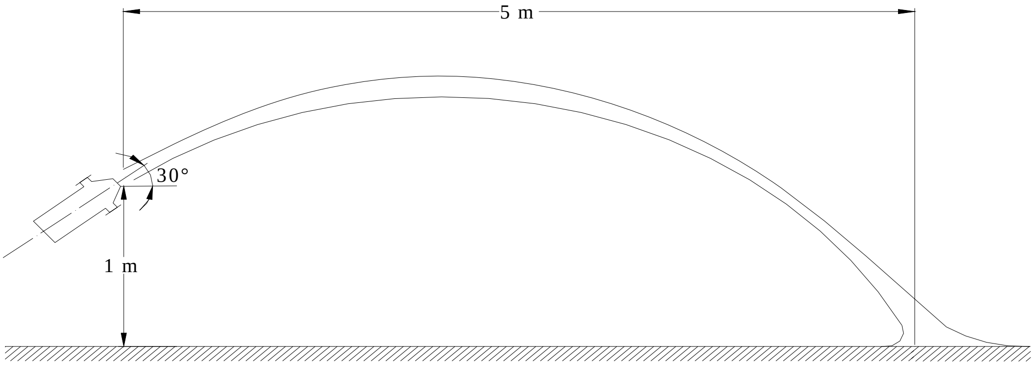


Fig. 4.11

SOLUTION:

$$(V_O)_x = V_j Cos30^0 = 0.866V_j$$
$$(V_O)_z = V_j Sin30^0 = 0.5V_j$$

Therefore the two coordinate equations for the trajectory are

$$x = (V_j Cos30^0)t$$

$$z = (V_j Sin30^0)t - \frac{1}{2}gt^2$$

By eliminating t and solving for V_j from these two equations,

$$V_j = \frac{x}{Cos30^0}\left[\frac{g}{2(xTan30^0 - z)}\right]^{1/2}$$
$$= \frac{5}{0.866}\left[\frac{9.81}{2(5\times 0.577 - 1)}\right]^{1/2}$$

$$V_j = 9.31 m/\text{sec}$$

Hence

$$Q = A_j V_j = \frac{\pi}{4}\times 0.05^2 \times 9.31 = 0.0183 m^3/\text{sec}$$

4.6.3. Venturimeter

If the flow constriction in a pipe, as shown in Fig. 4.12, is well streamlined, the loss of energy is practically equal to zero. The difference in velocity heads, $\Delta(V^2/2g)$, at two sections across the constriction results in a change in potential heads, $\Delta h_{pz}=\Delta(p/\gamma)+\Delta z$. Such a device is used for metering the quantity of flow Q in the pipe system by measuring the difference in potential head.

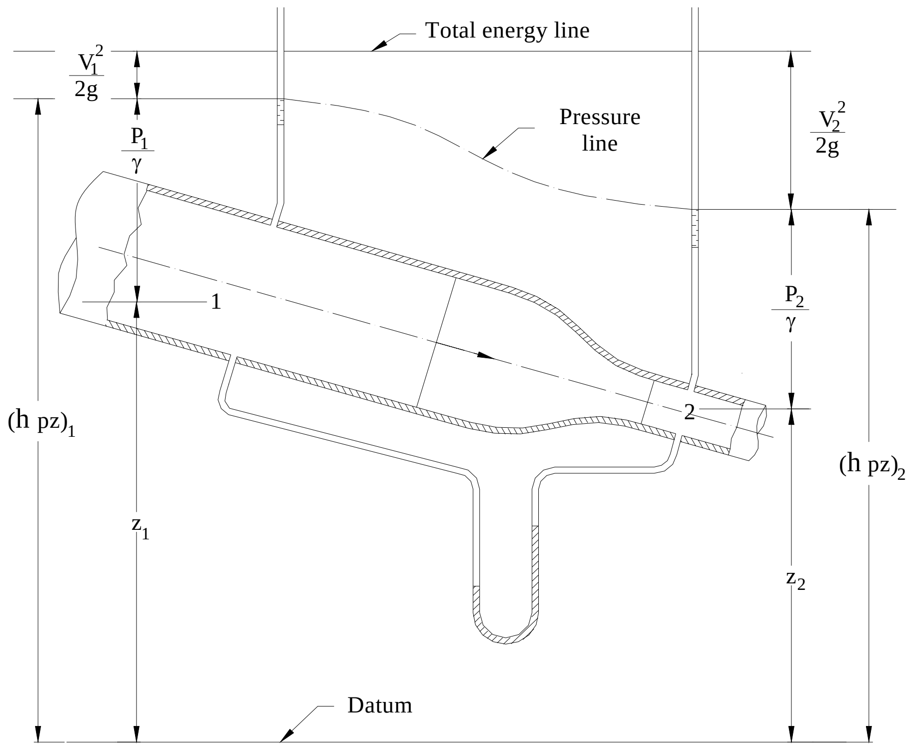


Fig. 4.12

The Bernoulli's equation for the two sections is

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

and the continuity equation is

$$Q = A_1 V_1 = A_2 V_2$$

These two equations may be solved for V_1 and V_2 it the two potential heads $(h_{pz})_1 = (p_1/\gamma) + z_1$ and $(h_{pz})_2 = (p_2/\gamma) + z_2$ are known. Hence, Q may be calculated. The difference in potential heads may be measured either by means of two piezometric columns at two sections or by using a differential manometer connected to the two sections.

EXAMPLE 4.5: The inclined Venturimeter shown in Fig. 4.13 is installed in a 20 cm (diameter) water pipe line and has a throat diameter of 10 cm. Water flows in the upward direction. For a manometer reading of 25 cm of mercury, what is the discharge in cubic meter per second? The specific weight of mercury is 13.6 gr/cm^3.

SOLUTION: Denote h as the vertical distance between the throat and the water-mercury interface in the manometer tube and y as the vertical distance between the centers of the two sections in which manometer tappings are located. If the friction loss of flow through the Venturimeter is neglected, Bernoulli's equation can be applied to sections A and t:

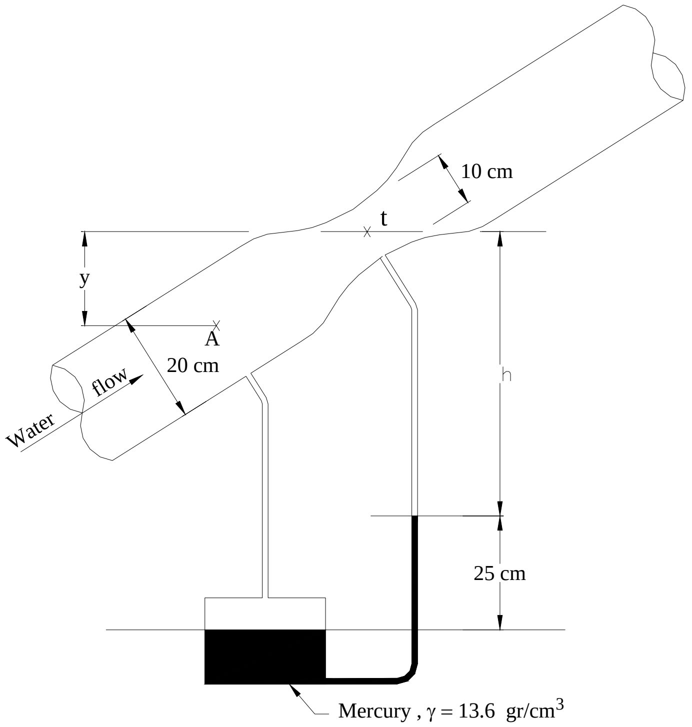


Fig. 4.13

$$\frac{p_A}{\gamma_w} + z_A + \frac{V_A^2}{2g} = \frac{P_t}{\gamma_w} + z_t + \frac{V_t^2}{2g}$$

By taking section A as the elevation datum,

$$\frac{p_A}{\gamma_w} + 0 + \frac{V_A^2}{2g} = \frac{P_t}{\gamma_w} + y + \frac{V_t^2}{2g} \qquad \text{(a)}$$

The difference in pressure head ($p_A/\gamma_w - p_t/\gamma_w$) is determined by writing pressure equation for the differential manometer starting at section A:

$$\frac{p_A}{\gamma_w} + (h - y + 0.25) - 0.25 \times 13.6 - h = \frac{p_t}{\gamma_w}$$

From this

$$\frac{p_A}{\gamma_w} - \frac{p_t}{\gamma_w} = y + 0.25 \times 13.6 - 0.25$$

Which now is substituted into Equ. (a) to obtain,

$$\frac{V_t^2}{2g} - \frac{V_A^2}{2g} = 0.25 \times 13.6 - 0.25 = 3.15m \qquad \text{(b)}$$

The continuity equation of flow is

$$A_A V_A = A_t V_t$$

$$\frac{\pi}{4}\times 0.2^2 \times V_A = \frac{\pi}{4}\times 0.1^2 \times V_t$$

$$V_A = \frac{1}{4}V_t$$

Then, by substituting $V_t/4$ for V_A in Equ. (b),

$$\frac{V_t^2}{2g}\left(1-\frac{1}{16}\right) = 3.15$$

Hence the velocity at the throat

$$V_t = \sqrt{\frac{2g\times 3.15\times 16}{15}} = 8.12\, m/\text{sec}$$

The discharge is

$$Q = A_t V_t = \frac{\pi}{4}\times 0.10^2 \times 8.12 = 0.064\, m^3/\text{sec}$$

4.6.4. Stagnation Tube

A stagnation tube, such as the one shown in Fig. 4.14, is simply a bent tube with its opening pointed upstream toward the approaching flow. The tip of the stagnation tube is *stagnation point*, and stagnation tube, therefore measures the *stagnation pressure* (or the total pressure), which is the sum of *static pressure* and the *dynamic pressure*.

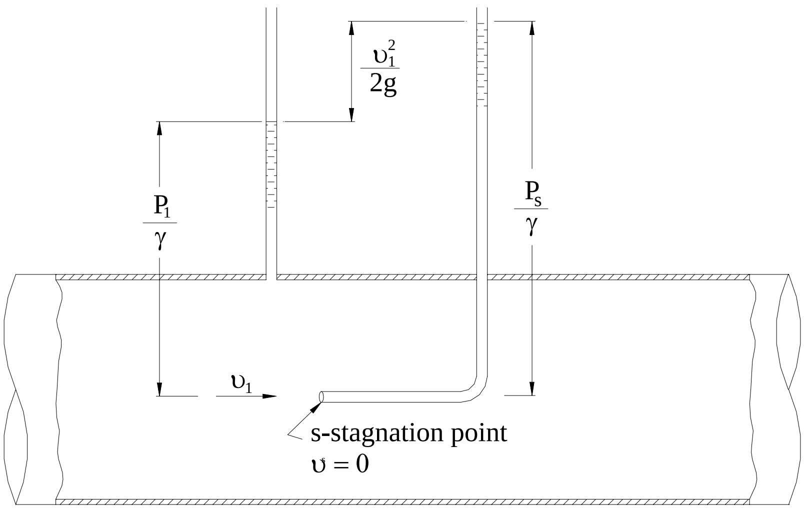


Fig. 4.14

By using the Bernoulli equation and taking the stagnation point s be section 2 as datum, the stagnation pressure p_s is determined:

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}=\frac{p_s}{\gamma}+0$$

or

$$p_s = p_1 + \frac{1}{2}\rho V_1^2 \qquad (4.11)$$

Stagnation pressure = Static pressure + Dynamic pressure

4.6.5. Pitot Tube

A typical *Pitot tube* is shown schematically in Fig. 4.15. It consists of a stagnation tube surrounded by a closed outer (static pressure) tube with annular space in between them. Small holes are drilled through the outer tube to measure the static pressure. The stagnation tube in the center measures the stagnation (total) pressure which is the sum of the static and the dynamic pressure.

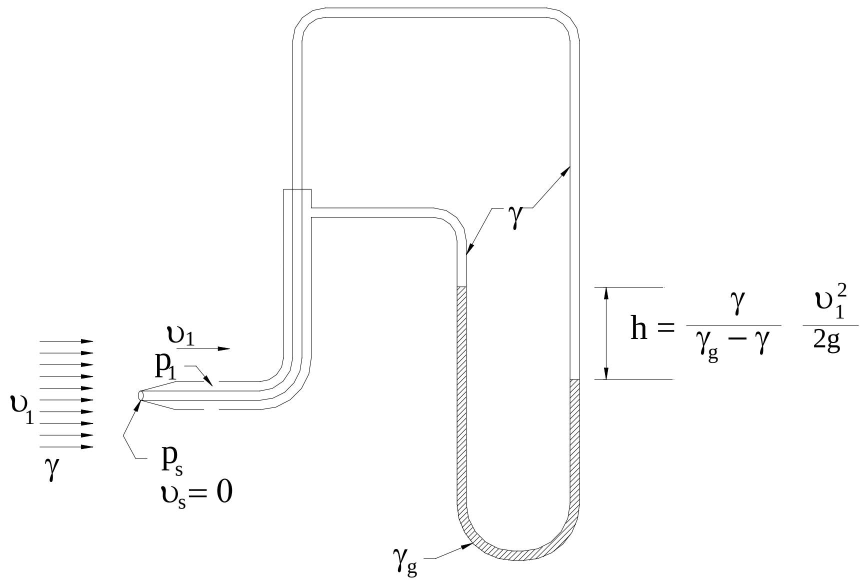


Fig. 4.15

When the two tubes are connected to a differential pressure-measuring device, the resulting difference in the pressure, p_s-p_1, is a direct measure of the velocity V_1:

$$V_1 = \sqrt{\frac{2(p_s - p_1)}{\rho}} \qquad (4.12)$$

EXAMPLE 4.6: The Pitot tube in Fig. 4.16 is carefully aligned with an air stream of specific weight 1.23 kg/m^3. If the attached differential manometer shows a reading of 150 mm of water, what is the velocity of the air stream?

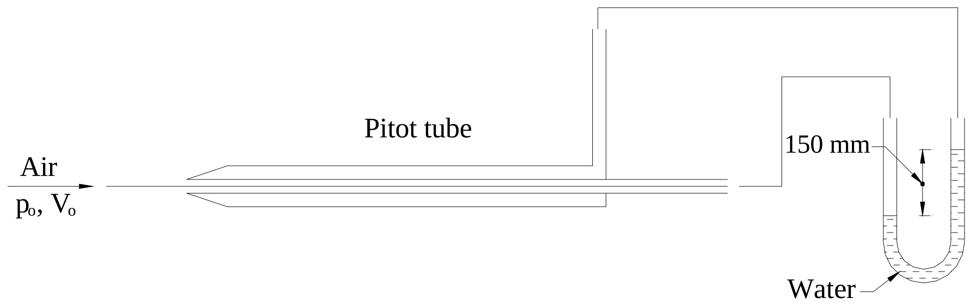


Fig. 4.16

SOLUTION: Stagnation pressure will be found at the tip of the Pitot tube. Assuming that the holes in the barrel of the static tube will collect the static pressure p_0 in the undisturbed air stream, the manometer will measure (p_s-p_0). Applying Equ. (4.11),

$$p_s = p_0 + \frac{1}{2}\rho V_0^2$$

Therefore,

$$0.15(1 - 0.00123) = \frac{1}{2} \times \frac{0.00123}{9.81} \times V_0^2$$
$$V_0 = 48.90\, m/\mathrm{sec}$$

4.6.6. Flow Over a Weir

The Bernoulli principle may be applied to problems of open flow such as the overflow structure of Fig. 4.17. Such problems feature a moving liquid surface in contact with the atmosphere and flow pictures dominated by gravitational action. A short distance upstream from the structure, the streamlines will be straight and parallel and the velocity distribution will be uniform.

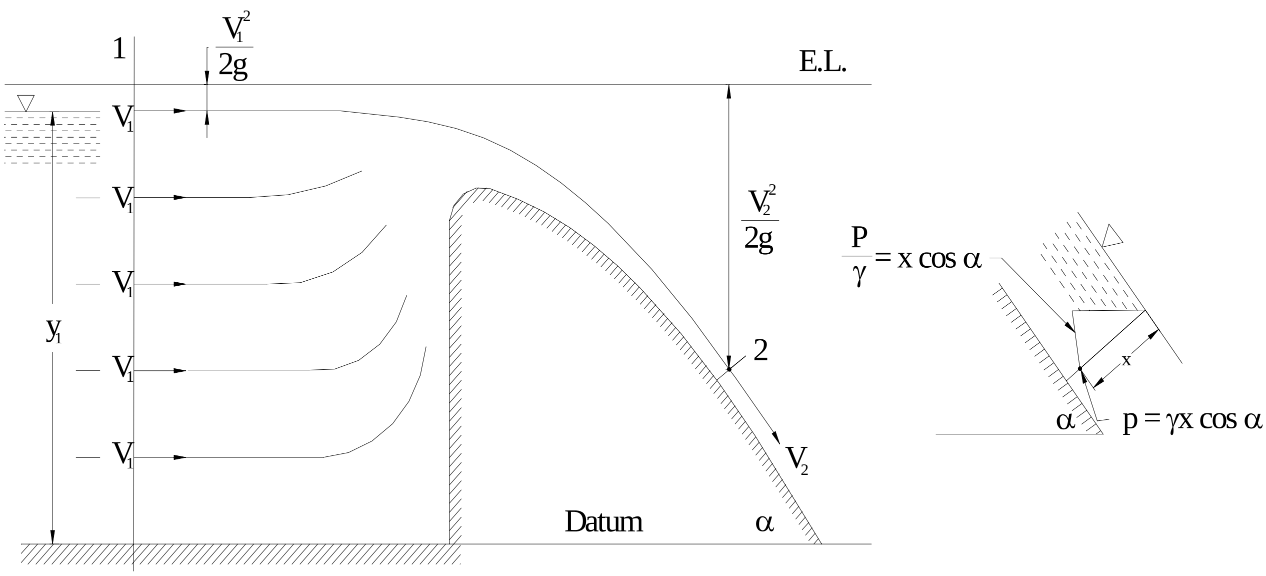


Fig. 4.17

In this region the quantity z+p/γ will be constant, the pressure distribution hydrostatic, and the hydraulic grade (piezometric) line (for all stream tube) located in the liquid surface; the energy line will be horizontal and located $V_1^2/2g$ above the liquid surface. With atmospheric pressure on the liquid surface the stream tube in the liquid surface behaves as a free jet allowing all surface velocities to be computed from the positions of liquid surface and energy line. The prediction of velocities elsewhere in the flow field where stream tubes are severely convergent or curved is outside the province of one-dimensional flow. At section 2, however (if the streamlines there are assumed straight and parallel), the pressures and velocities may be computed from the one-dimensional assumption.

EXAMPLE 4.7: Refer to Fig. 4.17. At section 2 the water surface is at elevation 30.5 m, and the 60^0 spillway surface at elevation 30 m. The velocity in the water surface V_{S2} at section 2 is 6.1 m/sec. Calculate the pressure and velocity on the spillway face at section 2. If the bottom of the approach channel is at elevation 29 m, calculate the depth and velocity in the approach channel.

SOLUTION:

Thickness of sheet of water at section 2 = $\dfrac{30.5-30}{Cos60^0}=1m$

Pressure on spillway face at section 2 = $1\times 1\times Cos60^0 0.5t/m^2$

Elevation of energy line = $30.5+\dfrac{6.1^2}{2g}=32.40m$

$$32.4=\frac{0.5}{1}+\frac{V_{F2}^2}{2g}+3.0$$

$$V_{F2}=6.1m/\sec$$

Which is to be expected from the one-dimensional assumption. Evidently all velocities through section 2 are 6.1 m/sec, so

$q=1\times 1\times 6.1=6.1m^3/\sec$ per meter of spillway length

At section 1,

$$y_1+29.0=z_1+\frac{p_1}{\gamma}$$

and applying the Bernoulli equation,

$$y_1+\frac{V_1^2}{2g}=y_1+\left(\frac{1}{2g}\right)\times\left(\frac{6.1}{y_1}\right)^2=3.4m$$

Solving this cubic equation by trial and error, the roots are y_1 = 3.22 m, 0.85 m and -0.69 m. Obviously the second and third roots are invalid here, so depth in approach channel will be 3.22 m. The velocity V_1 may be computed from

$$\frac{V_1^2}{2g} = 3.40 - 3.22 = 0.22m$$

or from

$$V_1 = \frac{6.10}{3.22}$$

Both of which give V_1 = 1.9 m/sec.

4.1.6. The Power of a Stream of Fluid

A stream of fluid could do work as a result of its pressure p, velocity V and elevation z and that the total energy per unit weight H of the fluid is given by

$$H = \frac{p}{\gamma} + \frac{V^2}{2g} + z$$

If the weight per unit time of fluid is known, the power of the stream can be calculated, since

Power = Energy Per unit time = (Weight/Unit time)×(Energy/Unit weight)

If Q is the volume rate (discharge) of flow,

Weight per unit time $= \gamma Q = \rho g Q$

Power $= \rho g Q H = \rho g H \left(\frac{p}{\gamma} + \frac{V^2}{2g} + z \right)$

Power $= pQ + \frac{1}{2} \rho V^2 Q + \rho g Q z$ (4.13)

EXAMPLE 4.8: Water is drawn from a reservoir, in which the water level is 240 m above datum, at the rate of 0.13 m^3/sec. The outlet of the pipeline is at datum level and is fitted with a nozzle to produce a high-speed jet to drive a turbine of the Pelton wheel type. If the velocity of the is 66 m/sec, calculate

a) The power of the jet,
b) The power supplied from the reservoir,
c) The head used to overcome losses,
d) The efficiency of the pipeline and nozzle in transmitting power.

SOLUTION:

a) The jet issuing from the nozzle will be at atmospheric pressure and at datum level so that, in Equ. (4.13), p = 0 and z = 0. Therefore,

Power of jet = $\frac{1}{2}\rho V^2 Q$

$$= \frac{1}{2} \times \frac{1000}{9.81} \times 66^2 \times 0.13 = 28862\, kgm/\sec$$
$$= 28862 \times 9.81 = 283140W = 283.14KW$$

b) At the reservoir, the pressure is atmospheric and the velocity of the free surface is zero so that, in Equ. (4.13), p = 0, V = 0. Therefore,

Power supplied from reservoir = $\rho Qgz = \gamma Qz$

$$= 1000 \times 0.13 \times 240 = 31200\, kgm/\sec$$
$$= 31200 \times 9.81 = 306072W = 306.72KW$$

c) If, H_1 = Total head at the reservoir, H_2 = Total head at the jet, h = Head lost in transmission,

Power supplied from reservoir = $\gamma QH_1 = 31200\, kgm/\sec$

Power of issuing jet = $\gamma QH_2 = 28862\, kgm/\sec$

Power lost in transmission = $\gamma Qh = 2338\, kgm/\sec$

Head lost in pipe = h = (Power lost)/(γQ)

$$h = \frac{2338}{1000 \times 0.13} = 17.98m$$

d) Efficiency of transmission = (Power of jet)/(Power supplied by reservoir)

Efficiency of transmission = $\frac{28862}{31200} = 0.925 = 92.5\%$

4.7. IMPULSE-MOMENTUM EQUATION: CONSERVATION OF MOMENTUM

The impulse-momentum equation for fluid flow can be derived from the well-known Newton's second law of motion. The resultant force $\vec{F}$ acting on a mass particle m is equal to the time rate of change of linear momentum of the particle:

$$\vec{F} = \frac{d\left(m\vec{V}\right)}{dt}$$

This law applies equally well to a system of mass particles. The internal forces between any two mass particles of the system exists in pairs. They are both equal and opposite of each other and, therefore, will cancel out as set forth in Newton's third law of motion. The external forces acting on mass particles of the system can then be summed up and equated to the time of change of the linear momentum of the whole system,

$$\sum \vec{F} = \frac{\sum d\left(m\vec{V}\right)}{dt} \tag{4.13}$$

If we define momentum by $\vec{I} = m\vec{V}$, then

$$\sum \vec{F} = \frac{\sum d\vec{I}}{dt} \tag{4.14}$$

Extending Newton's second law of motion to fluid flow problem, take as a free body the fluid mass included between sections 1-1 and 2-2 within a length of a flow channel (Fig. 4.18).

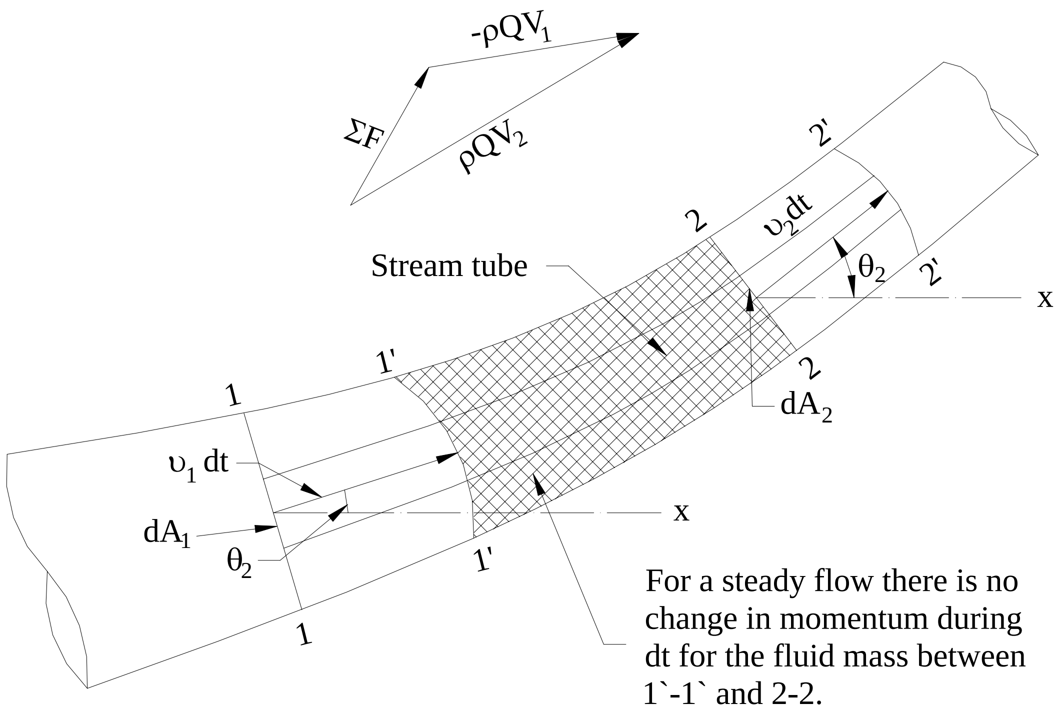


Fig. 4.18

The fluid mass of the free body 1-1 and 2-2 at time t_1 moves to a new position 1'-1' and 2'-2' at time t_2 when t_2-t_1 equals to dt. Sections 1'-1' and 2'-2' are curved because the velocities of flow at these sections are *non-uniform*. It should be noted that, for steady flow, the following continuity equation holds.

[Fluid mass within section 1-1 and 1'-1'] = [Fluid mass within section 2-2 and 2'-2']

$$M(11'1'1) = M(22'2'2)$$

The momentum of the system at time t,

$$\vec{I}_t = \vec{I}(11'1'1)_t + \vec{I}(1'221')_t$$

$$\vec{I}_t = \rho u_1 dA_1 dt \vec{u}_1 + \vec{I}(1'221')_t \tag{4.15}$$

The momentum of the system at time t+dt,

$$\vec{I}_{t+dt} = \vec{I}(1'221')_{t+dt} + \vec{I}(22'2'2)_{t+dt}$$

$$\vec{I}_{t+dt} = \vec{I}(1'221')_{t+dt} + \rho u_2 dA_2 dt \vec{u}_2 \tag{4.16}$$

Change of momentum at dt time,

$$d\vec{I} = \vec{I}_{t+dt} - \vec{I}_t \tag{4.17}$$

Furthermore, when the flow is steady,

$$\vec{I}(1'221')_t = \vec{I}(1'221')_{t+dt}$$

From Eqs. (4.15) and (4.16),

$$d\vec{I} = \rho u_2 dA_2 dt \vec{u}_2 - \rho u_1 dA_1 dt \vec{u}_1$$

or

$$\frac{d\vec{I}}{dt} = \rho u_2 dA_2 \vec{u}_2 - \rho u_1 dA_1 \vec{u}_1 \tag{4.18}$$

From Eqs. (4.14) and (4.18),

$$\sum \vec{F} = \rho u_2 dA_2 \vec{u}_2 - \rho u_1 dA_1 \vec{u}_1 \tag{4.19}$$

This equation is known as *impulse-momentum equation*.

If, however, average velocities V_1 and V_2 at sections 1-1 and 2-2 may be determined, the impulse-momentum equation may be written,

$$\sum \vec{F} = \rho V_2 dA_2 \vec{V}_2 - \rho V_1 dA_1 \vec{V}_1 \tag{4.20}$$

By taking integral over the areas,

$$\int \sum \vec{F} = \rho V_2 \vec{V}_2 \int_{A_2} dA_2 - \rho V_1 \vec{V}_1 \int_{A_1} dA_1$$

$$\vec{K} = \rho V_2 A_2 \vec{V}_2 - \rho V_1 A_1 \vec{V}_1 \tag{4.21a}$$

Here, $\vec{K} = \int \sum \vec{F}$ is the total of the external forces acting on the control volume.

For a steady flow of an incompressible fluid, the impulse-momentum equation for fluid flow may be simplified to the following form by first applying the continuity principle, that is, $Q = A_1V_1 = A_2V_2$, to the flow system:

$$\vec{K} = \rho Q\left(\vec{V}_2 - \vec{V}_1\right) \tag{4.21b}$$

If we take the velocity components for x and y-axes, Equ. (4.21b) takes the form of,

$$\begin{aligned} K_x &= \rho Q\left(V_{2x} - V_{1x}\right) \\ K_y &= \rho Q\left(V_{2y} - V_{1y}\right) \end{aligned} \tag{4.22}$$

These components can be combined to give the resultant force,

$$\vec{K} = \sqrt{K_x^2 + K_y^2} \tag{4.23}$$

If D'Alembert's principle is applied to the flow system, the system is brought into relative equilibrium with the inclusion of inertia forces. The value of K is positive in the direction if V is assumed to be positive.

For any control volume, the total force K that acts upon it in a given direction will be made up of three component forces:

$\vec{K}_1 =$ Force exerted in the given direction on the fluid in the control volume by any solid body within the control volume or coinciding with the boundaries of the control volume.

$\vec{K}_2 =$ Force exerted in the given direction on the fluid in the control volume by body forces such as gravity. (Weight of the control volume).

$\vec{K}_3 =$ Force exerted in the given direction on the fluid in the control volume by the fluid outside the control volume. (Pressure forces, always gage pressures were taken).

$\rho Q\vec{V}_1 =$ Rate of change of momentum in the direction of $\vec{V}_1$.

$\rho Q\vec{V}_2 =$ Rate of change of momentum in the opposite direction of $\vec{V}_2$.

Thus,

$$\vec{K} = \vec{K}_1 + \vec{K}_2 + \vec{K}_3 = \rho Q\left(\vec{V}_2 - \vec{V}_1\right) \tag{4.24}$$

The force F exerted by the fluid on the solid body inside or coinciding with the control volume in the given direction will be equal and opposite to $\vec{K}_1$ so that $\vec{F} = -\vec{K}_1$. The directions of the forces acting on a control volume can be shown schematically in Fig. (4.19).

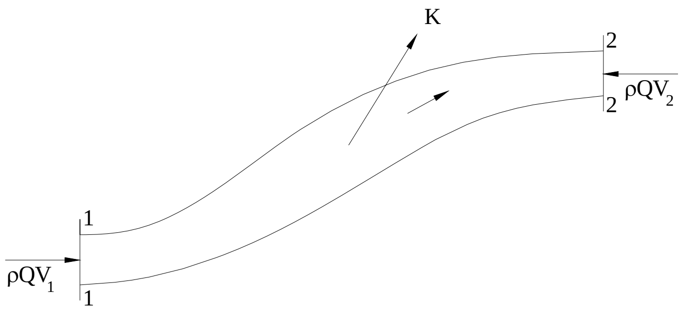


Fig. 4.19

4.7.1. Momentum Correction Factor

The momentum equation (4.24) is based on the assumption that the velocity is constant across any given cross-section. When a real fluid flows past a solid boundary, shear stresses are developed and the velocity is no longer uniform over the cross-section. In a pipe, for example, the velocity will vary from zero at the wall to a maximum at the center. Then, Equ. (4.24) takes the form of,

$$\vec{K} = \rho Q\left(\beta_2\vec{V}_2 - \beta_1\vec{V}_1\right)$$

Dimensionless *momentum correction factor* β accounts for the non-uniform distribution of velocity across the flow section. Obviously,

$$\beta = \frac{1}{AV^2}\int_A v^2 dA \tag{4.26}$$

and the numerical value of β is always greater one. In problems of turbulent flow in pipes, β is approximately equal to one.

Thus, Equ. (4.25) takes the form of Equ. (4.24) for the most practical problems of turbulent flow.

$$\vec{K} = \rho Q\left(\vec{V}_2 - \vec{V}_1\right) \tag{4.24}$$

4.7.2. Application of the Impulse-Momentum Equation

The impulse-momentum equation, together with the energy equation, and the continuity equation, furnishes the basic mathematical relationships for solving various engineering problems in fluid mechanics. In contrast to the energy equation, which is a *scalar* equation, each term in the impulse-momentum equation represents a *vector* quantity. The energy equation describes the conservation of energy and the average changes in the energies of flow along a flow passage, whereas the impulse-momentum equation relates the over-all forces on the boundaries of a chosen region in a flow channel without regard to the internal flow phenomenon. In many instances, however, both the impulse-momentum equation and the energy equation are complementary to each other.

Since the impulse-momentum equation relates the resultant external forces on a chosen free body of fluid in a flow channel to the change of momentum flux at the two end sections, it is especially valuable in solving those problems in fluid mechanics in which detailed information on the flow process may be either lacking or rather difficult to evaluate. In general, the impulse-momentum equation is used to solve the following two types of flow problems.

1) To determine the resultant forces exerted on the boundaries of a flow passage by a stream of flow as the flow *changes its direction or its magnitude of velocity or both*. Problems of this type include pipe bends and reducers, stationary and moving vanes, and jet propulsion. In such cases, although the fluid pressures on the boundaries may be determined by means of the energy equation, the resultant forces on the boundaries must be determined by integrating the pressure forces.
2) To determine the flow characteristics of non-uniform flows in which *an abrupt change of flow section occurs*. Problems of this type, such as a sudden enlargement in a pipe system or a hydraulic jump in an open-channel flow, cannot be solved by using the energy equation alone, because there is usually an unknown quantity of energy loss involved in each of these flow processes. The impulse-momentum equation must be used first to determine the flow characteristics. Then the energy equation may be used to evaluate the amount of energy loss in the flow process.

4.7.1.1. Force Exerted by a Flowing Fluid on a Contracting Pipe on a Horizontal Plane

The change of momentum of a fluid flowing through a pipe bend induces a force on the pipe. Consider the pipe bend shown in Fig. 4.20.

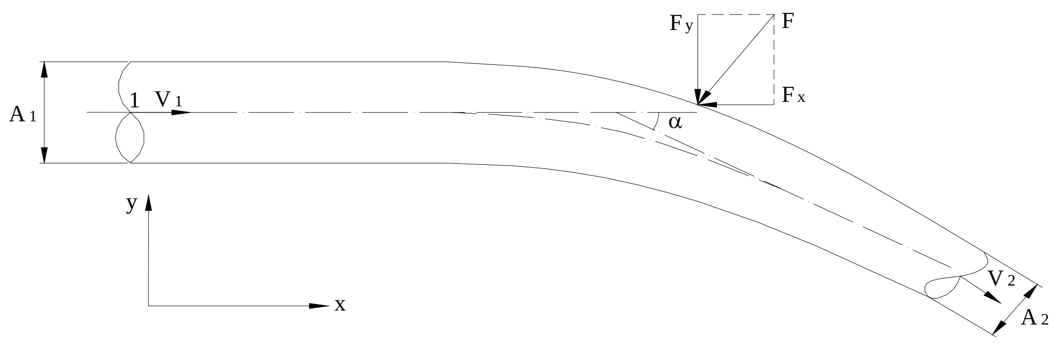


Fig. 4.20

The fluid enters the bend with velocity V_1 through area A_1 and leaves with a velocity V_2 through area A_2 after having been turned through an angle α. Let F be the force required to hold the pipe in equilibrium against the pressure of the fluid. F_x and F_y are the component forces in the negative x and y directions, respectively. Equ. (4.24) can now be employed to determine the magnitude of the force F. Along the x-direction we have,

$$\rho Q\left(V_2 Cos\alpha - V_1\right) = -F_x + p_1 A_1 - p_2 A_2 Cos\alpha \qquad \text{(a)}$$

and in the y-direction we have,

$$\rho Q\left(-V_2 Sin\alpha + 0\right) = -F_y + p_2 A_2 Sin\alpha \qquad \text{(b)}$$

Then the total force is given by,

$$F = \sqrt{F_x^2 + F_y^2} = \begin{cases} \rho^2 Q^2\left(V_1^2 + V_2^2 - 2V_1V_2 Cos\alpha\right) + p_1^2 A_1^2 + p_2^2 A_2^2 - \\ 2p_1p_2A_1A_2Cos\alpha + 2\rho Q\left[p_1V_1A_1 + p_2V_2A_2 - \left(p_2V_1A_2 + p_1V_2A_1\right)Cos\alpha\right] \end{cases}^{1/2} \qquad \text{(c)}$$

But, from the continuity equation the velocities are related by,

$$V_2 = \frac{A_1}{A_2}V_1$$

Thus, Equ. (c) becomes,

$$F = \left\{ \begin{aligned} &\rho^2 Q^2 V^2\left(1 - 2\frac{A_1}{A_2}Cos\alpha + \frac{A_1^2}{A_2^2}\right) + 2\rho Q^2\left[p_1 + p_2 - \left(p_1\frac{A_1}{A_2} + p_2\frac{A_2}{A_1}\right)Cos\alpha\right] \\ &+ p_1^2 A_1^2\left[1 + \left(\frac{p_2}{p_1}\right)^2\left(\frac{A_2}{A_1}\right)^2 - 2\left(\frac{p_2}{p_1}\right)\left(\frac{A_2}{A_1}\right)Cos\alpha\right] \end{aligned} \right\}^{1/2} \qquad \text{(d)}$$

For equal areas ($A_1 = A_2$), Equ. (d) reduces to,

$$F = \left\{2\rho Q^2\left(1 - Cos\alpha\right)\left(\rho V_1^2 + p_1 + p_2\right) + p_1^2 A_1^2\left[1 + \left(\frac{p_2}{p_1}\right)^2 - 2\left(\frac{p_2}{p_1}\right)Cos\alpha\right]\right\}^{1/2}$$

and, if the bend is 90^0, the force for the constant area bend becomes,

$$F = \left\{2\rho Q^2\left(\rho V_1^2 + p_1 + p_2 +\right) + p_1^2 A_1^2\left[1 - \left(\frac{p_1}{p_2}\right)^2\right]\right\}^{1/2}$$

EXAMPLE 4.8: When 300 lt/sec of water flow through this vertical 300 mm by 200 mm pipe bend, the pressure at the entrance is 7 t/m². Calculate the force by the fluid on the bend if the volume of the bend is 0.085 m³.

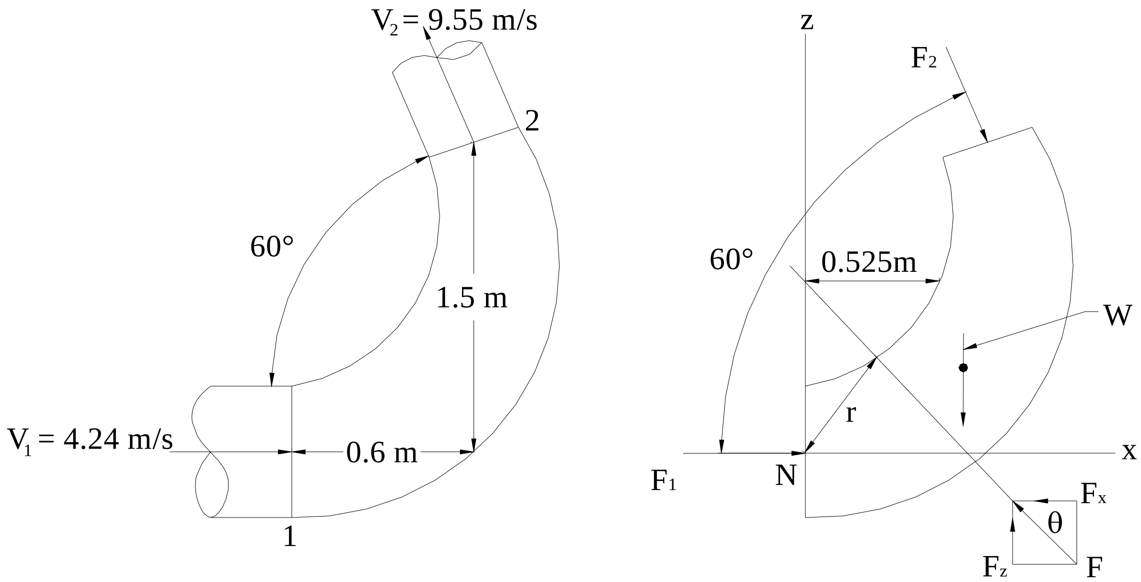


Fig. 4.21

SOLUTION: From the continuity principle,

$$Q = V_1 A_1 = V_2 A_2$$

$$V_1 = \frac{4Q}{\pi D_1^2} = \frac{4 \times 0.3}{\pi \times 0.3^2} = 4.24\, m/\sec$$

$$V_2 = \frac{4Q}{\pi D_2^2} = \frac{4 \times 0.3}{\pi \times 0.2^2} = 9.55\, m/\sec$$

and from the Bernoulli equation,

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$7 + 0 + \frac{4.24^2}{19.62} = \frac{p_2}{\gamma} + 1.5 + \frac{9.55^2}{19.62}$$

$$\frac{p_2}{\gamma} = 1.77 m \quad , \ p_2 = 1.77\, t/m^2$$

Now, for the free-body diagram, the pressure forces F_1 and F_2 may be computed,

$$F_1 = \frac{\pi}{4} \times 0.3^2 \times 7 = 0.495 ton$$

$$F_2 = \frac{\pi}{4} \times 0.2^2 \times 1.77 = 0.056 ton$$

With this and velocity diagram,

$$\sum x = 0$$

$$F_x = F_1 + \rho Q V_1 + F_2 Cos60^0 + \rho Q V_2 Cos60^2$$

$$F_x = 0.495 + \frac{1}{9.81} \times 0.30 \times 4.24 + \left(0.056 + \frac{1}{9.81} \times 0.3 \times 9.55\right) Cos60^0$$

$$F_x = 0.799 ton$$

$$\sum z = 0$$

$$F_z = \left(F_2 + \rho Q V_2\right) Sin60^0 + \gamma \times 0.085$$

$$F_z = \left(0.056 + \frac{1}{9.81} \times 0.30 \times 9.55\right) Sin60^0 + 1 \times 0.085$$

$$F_z = 0.386 ton$$

$$F = \sqrt{F_x^2 + F_z^2} = \sqrt{0.779^2 + 0.386^2} = 0.869 ton$$

$$Tan\theta = \frac{F_z}{F_x} = \frac{0.386}{0.799} = 0.483 \quad , \ \theta = 25.8^0$$

The plus signs confirm the direction assumptions for F_x and F_z. Therefore the force on the bend is 0.87 ton downward to the right at 25.8^0 with horizontal.

Now, assuming that the bend is such shape that the centroid of the fluid therein is 0.525 m to the right of section 1 and that F_1 and F_2 act at the centroids of sections 1 and 2, respectively. We take moments about the center of section 1 to find the location of F,

$$-r \times 0.87 + 0.525 \times 0.085 + 1.5 \times 0.056 \times Cos60^0 + 0.6 \times 0.056 \times Sin60^0 =$$

$$-1.5 \times 9.55 \times Cos60^0 \times 0.30 \times \frac{1}{9.81} - 0.6 \times 9.55 \times Sin60^0 \times 0.30 \times \frac{1}{9.81}$$

$$r = 0.56 m$$

4.7.1.2. Reaction of a Jet

In Chapter (4.6.1) Torricelli's theorem for the efflux velocity from an orifice in a large tank was derived. The momentum theorem can now be applied to this example to determine the propulsive force created by the orifice flow. In Fig. 4.22 is shown a large tank with its surface open to the atmosphere and with an orifice area A_2.

We assume that $A_2 << A_1$. Then the velocity of the jet by Equ. (4.10),

$$V_2 = \sqrt{2gh}$$

and

$$Q = A_2 V_2$$

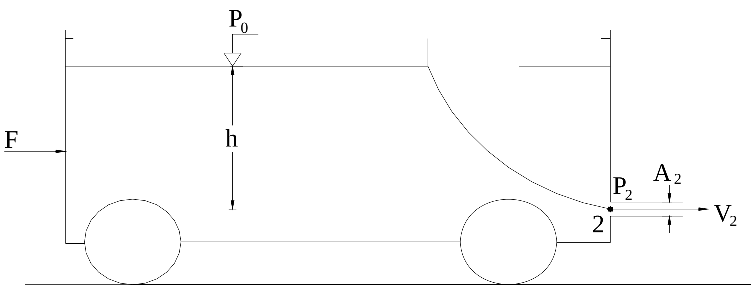


Fig. 4.22

Let F represent the force necessary to hold the tank in equilibrium. Then Equ. (4.22) becomes,

$$F = \rho Q (V_2 - 0)$$

Therefore, the propulsive force or thrust induced by the jet is given by,

$$T = -F = -\rho Q V_2$$

EXAMPLE 4.9: A jet of water of diameter d = 50 mm issues from a hole in the vertical side of an open tank which is kept filled with water to a height of 1.5 m above the center of the hole (Fig. 4.23). Calculate the reaction of the jet on the tank and its contents, a) When it is stationary, b) When it is moving with a velocity u = 1.2 m/sec in the opposite direction to the jet relative to the tank remains unchanged. In the latter case, what would be the work done per second?

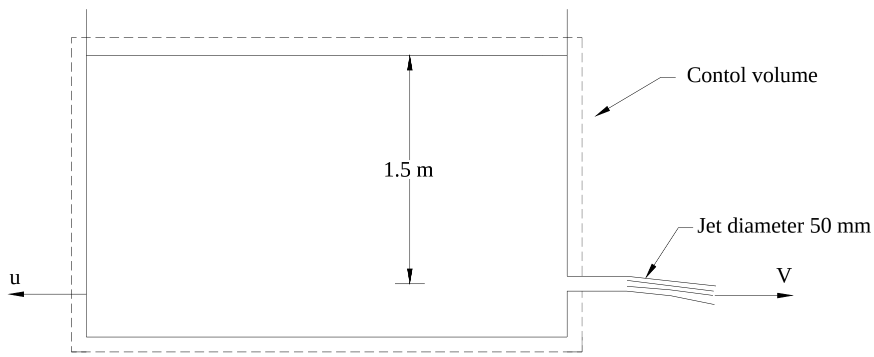


Fig. 4.23

SOLUTION: Take the control volume shown in Fig. 4.23. In Equ. (4.24), the direction under consideration will be that of the issuing jet, which will be considered as positive in the direction of the jet; therefore, $K_2 = 0$, and, if the jet is assumed to be at the same pressure as the outside of the tank, $K_3 = 0$. Force exerted by fluid system in the direction of motion,

$$R = -K_1 = -\rho Q (V_2 - V_1) \qquad \text{(a)}$$

The velocity of the jet may be found by applying Torricelli's equation (Equ. 4.10),

$$V_2 = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 1.5} = 5.42\, m/\text{sec}$$

Mass discharge per unit time,

$$M = \rho Q = \rho \frac{\pi d^2}{4} V_2 = \frac{1000}{9.81} \times \frac{\pi \times 0.05^2}{4} \times 5.42$$
$$M = 1.085\, kg \sec/m$$

a) If the tank is stationary,

V_2 = 5.42 m/sec

V_1 = Velocity at the free surface = 0

Substituting in Equ. (a), reaction of jet on the tank,

$$R = \rho Q V_2 = M V_2 = 1.085 \times 5.42 = 5.88 kg$$

b) If the tank is moving with a velocity u in the opposite direction to the of the jet, the effect is to superimpose a velocity of (–u) on the whole system:

$$V_2' = V_2 - u \qquad , V_1 = -u$$

Thus,

$$V_2 - V_1 = V_2$$

Thus, the reaction of the jet R remains unaltered at 5.88 kg.

Work done per second = Reaction × Velocity of the tank

$$W = R \times u = 5.88 \times 1.2 = 7.056\, kgm/\sec = 69.22 Watt$$

4.7.1.3.Pressure Exerted on a Plate by a Free Jet

Consider a jet of fluid directed at the inclined plate shown in Fig. 4.24. Suppose we are interested in equilibrium against the pressure of the jet.

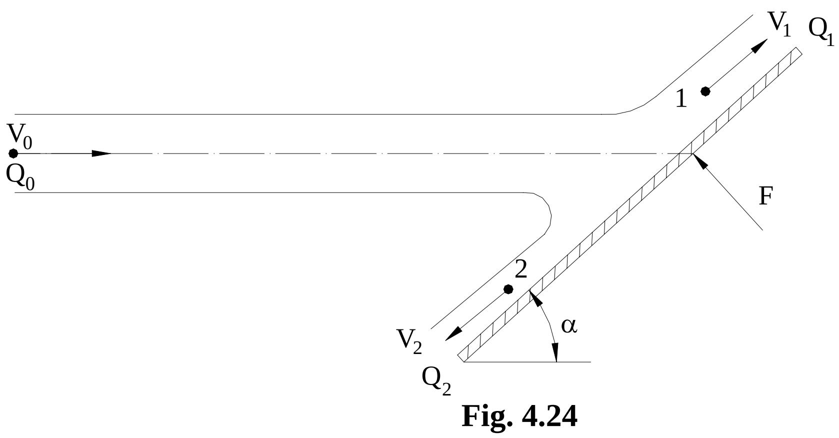


Fig. 4.24

For a free jet the static pressure is the same for all points in the jet. Thus the velocities can be related by,

$$V_0 = V_1 = V_2 \qquad \text{(a)}$$

In addition, for an inviscid fluid there can be no shearing force parallel to the plate surface; thus, the reaction force is normal to the plate surface. From the momentum theorem this force must be equal to the rate of momentum change normal to the plate surface. For this case,

$$F = \rho V_0 Q_0 Sin\alpha \qquad \text{(b)}$$

where $Q_0 = V_0 A$ and A is the cross-sectional area of the jet. It is interesting to note that the division of flow along the plate is uneven. The magnitudes of the flow rates along the plate can be determined by a consideration of the momentum theorem parallel to the plate. In this case,

$$(\rho Q_1 V_1 - \rho Q_2 V_2) - \rho Q_0 V_0 Cos\alpha = 0 \qquad \text{(c)}$$

Also, the continuity equation stipulates that,

$$Q_0 = Q_1 + Q_2 \qquad \text{(d)}$$

Combination of Eqs. (a), (c), and (d) leads to the following results for discharges:

$$Q_1 = \frac{Q_0}{2}(1 + Cos\alpha) \qquad \text{(e)}$$

$$Q_2 = \frac{Q_0}{2}(1 - Cos\alpha) \qquad \text{(f)}$$

EXAMPLE 4.10:

Calculate the force exerted by a jet of water 20 mm in diameter which strikes a flat plate at an angle of 30^0 to the normal of the plate with a velocity of 10 m/sec if, a) The plate is stationary, b) The plate is moving in the direction of the jet with a velocity of 2 m/sec.

SOLUTION:

a) The angle α shown in Fig. 4.24 is 90^0-30^0=60^0; hence the normal force [Equ.(b)] is,

$$F = \frac{1000}{9.81} \times \frac{\pi \times 0.02^2}{4} \times 10 \times 10 \times Sin60^0 = 2..77kg$$

b) The change of velocity on impact in this case is,

$$V_0 - V_p = 8\, m/\sec$$

The normal force is,

$$F = \frac{1000}{9.81} \times \frac{\pi \times 0.02^2}{4} \times 8 \times 8 \times Sin60^0 = 1.77 kg$$

4.7.1.4. Stationary and Moving Vanes: The Impulse Turbine

Fig. 4.25 shows a free jet, which is deflected by a stationary curved vane through an angle α. Here the jet is assumed to impinge on the vane tangentially. Hence, there is no loss of energy because of impact. Since the friction loss of the flow passing along the smooth surface of the stationary vane is almost equal to zero, the magnitude of the jet velocity remains unchanged as it flows along the vane if the small difference in elevation between the two ends of the vane is neglected as was shown in [Equ. (c). Pressure exerted on a plate by a free jet].

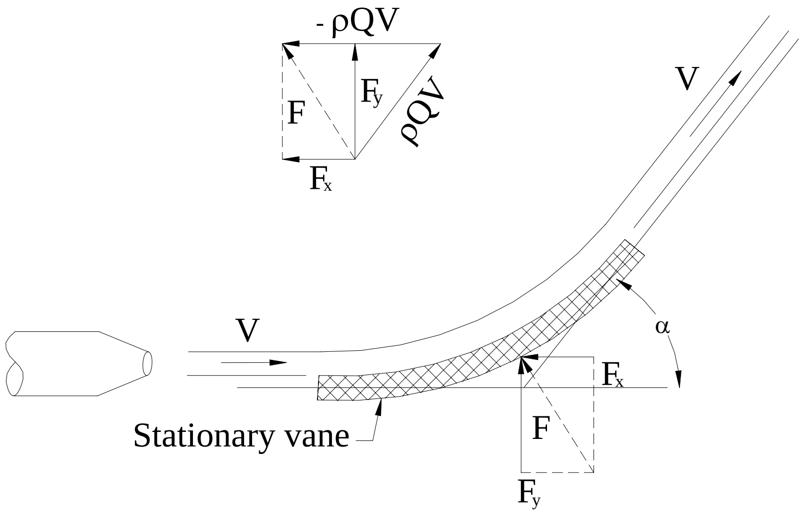


Fig. 4.25

Therefore, the two components F_x and F_y of the F exerted by the stationary vane on the jet of fluid may be determined from the following impulse-momentum equations:

$$-F_x = \rho QVCos\alpha - \rho QV$$

$$F_y = \rho QVSin\alpha - 0$$

The force components, which exerts on the vane are equal and opposite to F_x and F_y shown in Fig. 4.25.

Next, consider the moving vane in Fig. 4.26, which is moving with a velocity u in the same direction as the approaching jet. The free jet of velocity V hits the moving vane tangentially. This type of problem may be analyzed by applying the principle of relative motion to the whole system.

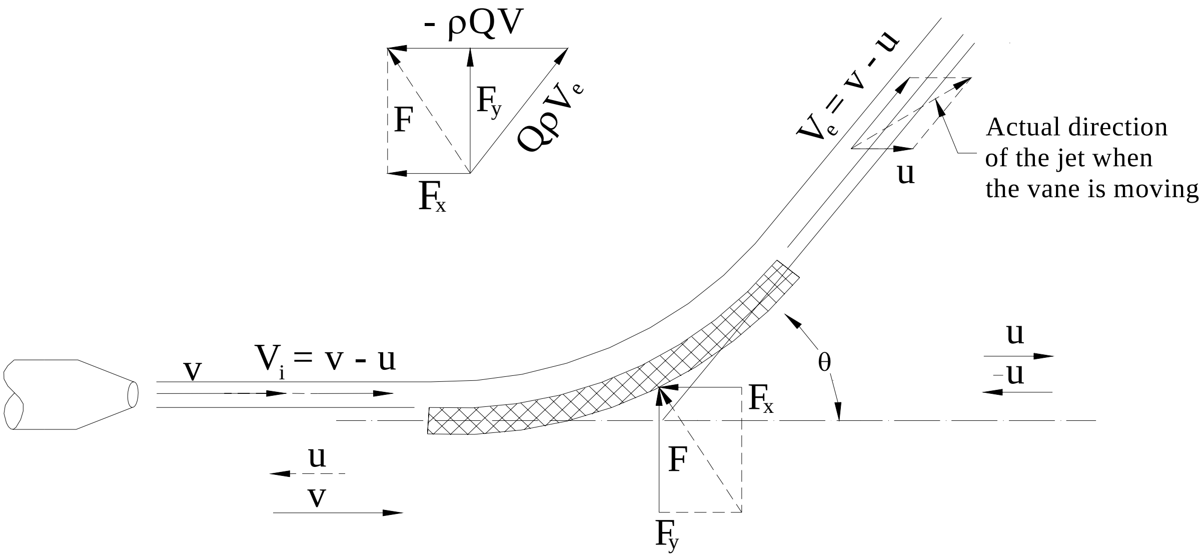


Fig. 4.26

This is done by bringing the moving vane in a stationary state before the entrance V and the exit V_e must be relative velocities of the jet at these two sections with respect to vane. The entrance velocity of the jet relative to the vane is V_i = V-u, and the magnitude of this relative velocity remains the same along the curved surface of the vane if the friction loss is assumed to be zero. Thus, V_e = V-u, and the direction of the exit velocity relative to the vane is shown in Fig. 4.26. Therefore, the force components F_x and F_y exerted by the moving vane on the jet are determined by applying the impulse-momentum equations to the flow system:

$$-F_x = Q\rho(V-u)Cos\alpha - Q\left(\frac{V_i}{V}\right)\rho(V-u)$$

$$F_x = \rho Q(V-u)(1-Cos\alpha)$$

$$F_y = \rho Q(V-u)Sin\alpha - 0$$

Again, the force components which jet exerts on the moving vane must be equal and opposite to F_x and F_y shown in Fig. 4.26.

In mechanics the *power* developed by a working agent is defined as the rate at which work done by that agent. When a jet of fluid strikes a single moving vane of Fig. 4.26, the power developed is equal to $F_x \times u$, or

Power = $\rho Q(V-u)(1-Cos\alpha)u$

Force component F_y does not produce any power because there is no motion of the vane in the y direction.

In the engineering application of the principle of moving vane to an *impulse turbine wheel* (Fig. 4.27), a series of vanes is mounted on the periphery of a rotating wheel. The vanes are usually spaced that the entire discharge Q is deflected by vanes.

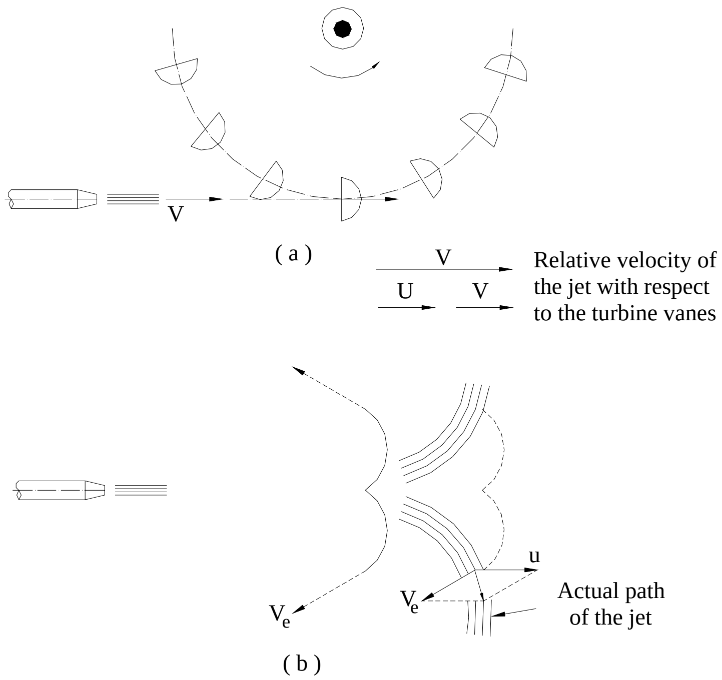


Fig. 4.27

Therefore, the total power output of a frictionless impulse turbine is,

$$P_T = \rho Q(V-u)(1-Cos\alpha)u \tag{4.25}$$

This equation indicates that, for any free jet of discharge Q and velocity V, the power developed in an impulse turbine is seen to vary with both the deflection angle α of the vane and the velocity u at which the vanes move. Mathematically, the values of α and u to produce maximum turbine power for a given jet may be determined by taking partial derivatives $\partial P_T/\partial\alpha$ and $\partial P_T/\partial u$ and then equating them zero. Thus,

$$\frac{\partial P_T}{\partial \alpha} = \rho Q(V-u)u(1+Sin\alpha) = 0$$

and

$$\alpha = 180^0 \quad ; \; (P_T)_{\alpha=180^0} = 2\rho Qu(V-u)$$

Also,

$$\frac{\partial P_T}{\partial u} = \rho Q(1-Cos\alpha)(V-2u) = 0$$

and,

$$u = \frac{V}{2} \qquad ; \; (P_T)_{u=V/2} = \rho Q(1-Cos\alpha)\frac{V^2}{4}$$

The *maximum turbine power* is obtained when α=180^0 and u=V/2.

Therefore,

$$(P_T)_{max} = \rho Q\frac{V^2}{2}$$

which is exactly the power in the free jet of fluid. In practice, however, the deflection angle of the vanes on an impulse wheel is found to be about 170 degrees and the periphery speed of the impulse wheel to be approximately u = 0.45V.

EXAMPLE 4.11: An impulse turbine of 1.8 m diameter is driven by a water jet of 50 mm diameter moving at 60 m/sec. Calculate the force on the blades and the power developed at 250 r/min. The blade angles are 150^0.

SOLUTION: The velocity of the impulse wheel is,

$$u = \frac{250}{60} \times 2\pi \times 0.9 = 23.6\, m/\sec$$

The flow rate is,

$$Q = \frac{\pi}{4} \times 0.05^2 \times 60 = 0.12\, m^3/\sec$$

The working component of force on the fluid is,

$$F_x = \rho Q (V - u)(1 - Cos\alpha)$$

$$F_x = \frac{1000}{9.81} \times 0.12 \times (60 - 23.6)(1 - Cos150^0)$$

$$F_x = 831 kg$$

The power developed,

$$P = F_x \times u = 831 \times 23.6 = 19612\, kgm/\sec = 192390 Watt$$

4.7.1.5. Sudden Enlargement in a Pipe System

There is a certain amount of energy lost when the fluid flows through a *sudden enlargement* in a pipe system such as that shown in Fig. 4.28.

The continuity equation for the flow is,

$$Q = A_1 V_1 = A_1 V_e = A_2 V_2$$

Since the velocity V_e of the submerged jet may be assumed to be equal to V_1, by reason of Bernoulli theorem, p_e equals to p_1. This latter condition is readily verified in the laboratory.

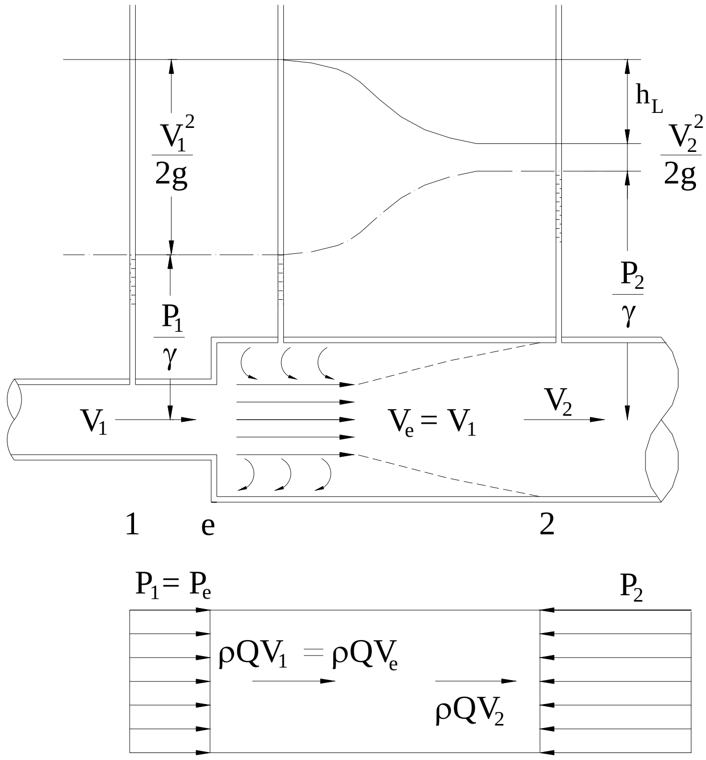


Fig. 4.28

The impulse-momentum equation for the flow of fluid in the pipe between sections e and 2 is,

$$(p_e - p_2)A_2 = \rho Q(V_2 - V_1)$$

or

$$\frac{p_1 - p_2}{\gamma} = \frac{Q}{gA_2}(V_2 - V_1) = \frac{V_1^2}{g}\left[\left(\frac{A_1}{A_2}\right)^2 - \frac{A_1}{A_2}\right]$$

The energy equation may now be written between sections 1 and 2 in the following form:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + h_L$$

Solving these three equations simultaneously gives,

$$h_L = \frac{(V_1 - V_2)^2}{2g} = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2 \qquad (4.26)$$

This is the well-known *Borda-Carnot* equation.

4.7.1.6. Hydraulic Jump in an Open Channel Flow

A *hydraulic jump* in an open channel flow is a local phenomenon in which the surface of a rapidly flowing stream of liquid rises abruptly. This sudden rise in liquid surface is accompanied by the formation of extremely turbulent rollers on the sloping surface in the hydraulic jump as that shown in Fig. 4.29. An appreciable quantity of energy is dissipated in this process when the initial kinetic energy of flow is partly transformed into potential energy.

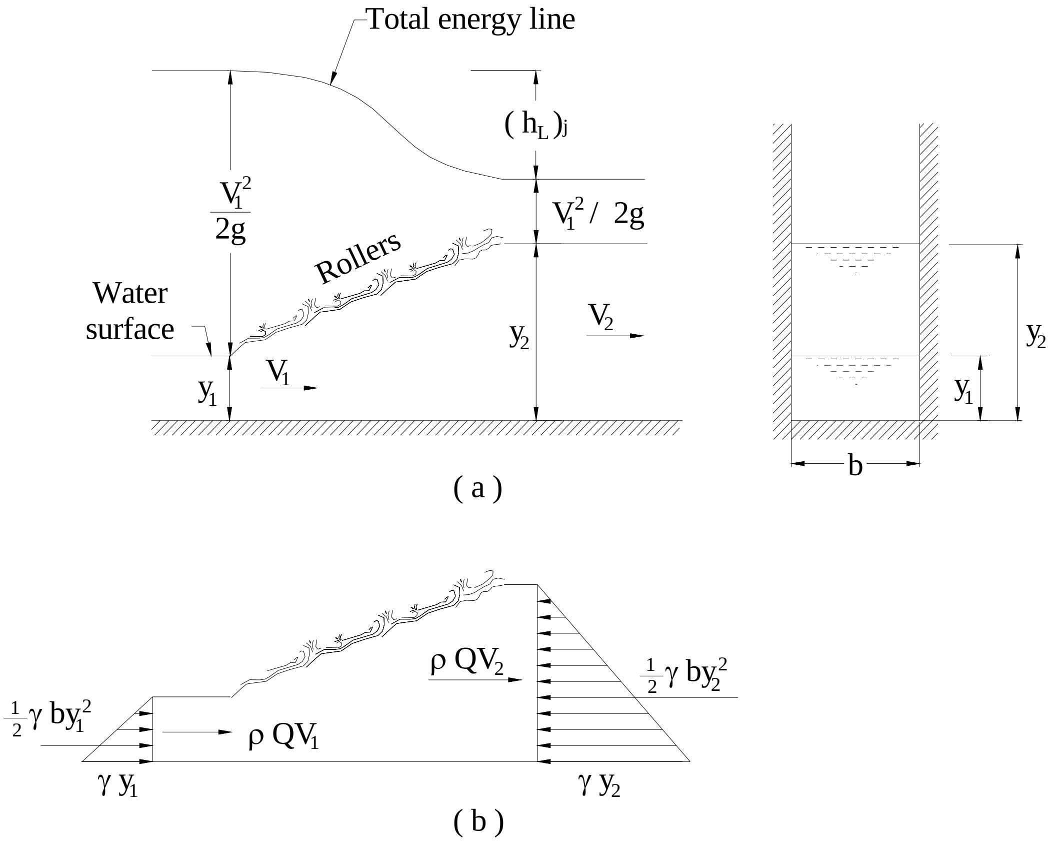


Fig. 4.29

The hydraulic jump shown in Fig. 4.29 is assumed to occur in a horizontal rectangular channel of width b (perpendicular to the plane of paper). Because the flow is guided by a solid boundary at the bottom of the channel, hydrostatic pressure distribution exists at both the upstream section 1 and downstream section 2 from the hydraulic jump. The following two assumptions are made in the mathematical analysis of a hydraulic jump: 1) The friction loss of flow at the wetted surface of the channel between sections 1 and 2 is assumed to be negligible; 2) The velocity distribution of flow at both sections is assumed to be uniform.

For steady flows the continuity equation yields,

$$Q = b\ y_1 V_1 = b y_2 V_2 \qquad \text{or} \qquad y_1 V_1 = y_2 V_2$$

and the impulse-momentum equation for the free body of the flow system (Fig.4.29b) is,

$$\frac{\gamma b y_1^2}{2} - \frac{\gamma b y_2^2}{2} = \rho Q V_2 - \rho Q V_1$$

or

$$\frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = \rho y_2 V_2^2 - \rho y_1 V_1^2$$

of the four quantities, V_1, y_1, V_2, and y_2, two must be given; the other two can be determined by the simultaneous solution of these two equations.

After the flow characteristics are all determined, the loss of energy of flow in a hydraulic jump can readily be evaluated by the application of the energy equation, that is,

$$h_L = \left(y_1 + \frac{V_1^2}{2g} \right) - \left(y_2 + \frac{V_2^2}{2g} \right) \qquad (4.27)$$

EXAMPLE 4.12: As shown in the Fig. 4.30, 10 m^3/sec of water per meter of width flows down an overflow spillway onto a horizontal floor. The velocity of flow at the toe of the spillway is 20 m/sec. Compute the downstream depth required at the end of the spillway floor to cause a hydraulic jump to form and the horsepower dissipation from the flow in the jump per meter of width.

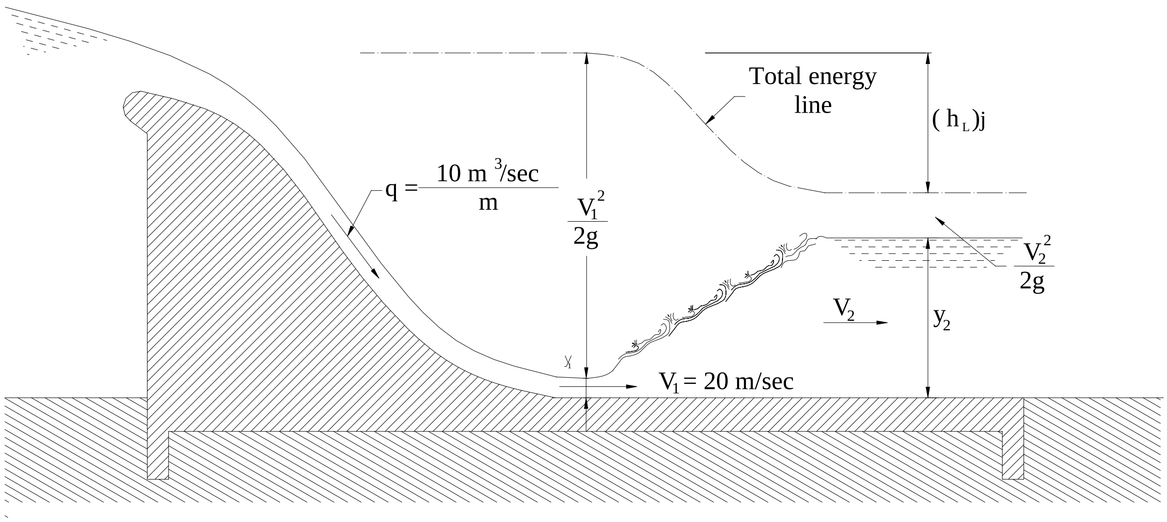


Fig. 4.30

SOLUTION: The continuity equation of flow is,

$$q = 10m^3/\text{sec}/m = 20y_1 = V_2 y_2$$

Hence,

$$y_1 = 0.5m$$

The momentum equation relating flow conditions at two end sections of the jump is,

$$\frac{1 \times 0.5^2}{2} - \frac{1 \times y_2^2}{2} = \frac{1}{9.81} \times y_2 \times V_2^2 - \frac{1}{9.81} \times 0.5 \times 20^2$$

Solving these equations simultaneously yield,

$V_2 = 1.63\, m/\sec$ and $y_2 = 6.14m$

From Equ. (4.27) the loss of energy of flow in the hydraulic jump is,

$$h_L = \left(0.5 + \frac{20^2}{2g}\right) - \left(6.14 + \frac{1.63^2}{2g}\right)$$

$h_L = 14.60m$ per meter width

Horsepower dissipation $= \dfrac{\gamma Q h_L}{75} = \dfrac{1000 \times 10 \times 14.60}{75} = 1946Hp$

CHAPTER 5

FLOW OF A REAL FLUID

5.1 INTRODUCTION

The flow of a real fluid is more complex than that of an ideal fluid, owing to the phenomena caused by the existence of viscosity. Viscosity introduces resistance to motion by causing shear and friction forces between fluid particles and boundary walls. For flow to take place, work must be done against resistance forces, and in the process energy is converted to heat. The inclusion of viscosity also allows the possibility of two physically different flow regimes. The effects of viscosity on the velocity profile also render invalid the assumption of uniform velocity distribution. Although the Euler equations may be altered to include shear stresses of a real fluid, the result is a set of partial differential equations to which no general solution is known.

5.2 REYNOLD'S EXPERIMENTS

The effects of viscosity cause the flow of a real fluid to occur under two different conditions or regimes: that of *laminar* flow and that of *turbulent* flow. The characteristics of these regimes were first demonstrated by Reynolds, with an apparatus similar to that of Fig. 5.1.

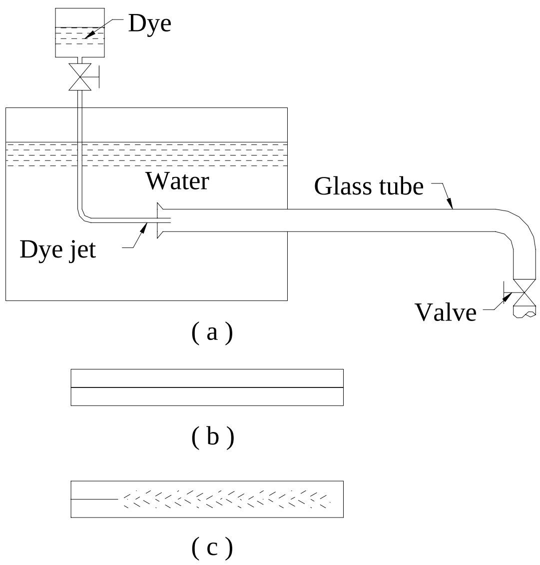


Fig. 5.1

Water flows from a tank through a bell-mouthed glass pipe, the flow being controlled by the valve. A thin tube leading from a reservoir of dye has its opening within the entrance of the glass pipe.

Reynolds discovered that, for low velocities of flow in the glass pipe, a thin filament of dye issuing from the tube did not diffuse but was maintained intact throughout the pipe, forming a thin straight line parallel to the axis of the pipe (Fig.5.1b). As the valve was opened, however, and greater velocities were attained, the dye filament wavered and broke, eventually diffusing through the flowing water in glass pipe (Fig.5.1c).

Since mixing of fluid particles during flow would cause diffusion of the dye filament, Reynolds deduced from his experiments that at low velocities this mixing was absent and that the fluid particles moved in parallel layers, or laminae, sliding past adjacent laminae but not mixing with them, this is the regime of *laminar flow*. Since at higher velocities the dye filament diffused through the pipe, it was apparent that mixing of fluid particles was occurring, or, in other words, the flow was *turbulent*. Laminar flow broke down into the turbulent flow at some *critical velocity* above that at which turbulent flow was restored to the laminar condition.

Reynolds was able to generalize his conclusions from his dye stream experiments by the introduction of a dimensionless term Re, later called the *Reynolds Number*, which was defined by

$$\mathrm{Re} = \frac{Vd\rho}{\mu} = \frac{Vd}{\nu} \tag{5.1}$$

In which V is the mean velocity of the fluid in pipe, d is the diameter of the pipe, and ρ, μ and ν are the specific mass, dynamic viscosity and kinematic viscosity of the fluid flowing therein.

The upper limit of laminar flow to be defined by $2100<\mathrm{Re}_{cr}<4000$. The lower limit of turbulent flow, defined by the lower critical Reynolds number, is of greater engineering importance; it defines a condition below which all turbulence entering the flow from any source will eventually be damped out by viscosity. This lower critical Reynolds number thus sets a limit below which laminar flow will always occur; many experiments have indicated the lower critical Reynolds number to have a value of approximately 2100. Between Reynolds number 2100 and 4000 a region of uncertainty exits.

The concept of a critical Reynolds number to the flow of any fluid in cylindrical pipes, one may predict that the flow will be laminar if Re<2000 and turbulent if Re>4000. However, critical Reynolds number is very much a function of boundary geometry.

Flow	**d length**	**Re_{cr}**
Flow in cylindrical pipes	pipe diameter	2000
Flow between parallel walls	Spacing between the walls	1000
Flow in a wide open channel	Water depth	500
Flow about a sphere	Sphere diameter	1

EXAMPLE 5.1: Water of kinematic viscosity 1.15×10^{-6} m²/sec flows in a cylindrical pipe of 30 mm diameter. Calculate the largest discharge for which laminar flow can be expected. What is the equivalent discharge for air? $\nu_{air} = 1.37\times10^{-5}$ m²/sec.

SOLUTION:

Taking Re_{cr} = 2100 as the upper limit for laminar flow,

$$\mathrm{Re}_{cr} = \frac{Vd}{\vartheta}$$

$$2100 = \frac{V \times 0.03}{1.15\times10^{-6}}$$

$$V = 0.0805 m/\sec$$

$$Q_{water} = 0.0805\times\frac{\pi}{4}\times0.03^2 = 5.69\times10^{-5}\ m^3/\sec$$

$$2100 = \frac{V_{air} \times 0.03}{1.37\times10^{-5}}$$

$$V_{air} = 0.959\, m/\sec$$

$$Q_{air} = 0.959\times\frac{\pi}{4}\times0.03^2 = 6.78\times10^{-4}\ m^3/\sec$$

$$Q_{air} \cong 12 Q_{water}$$

5.3. LAMINAR AND TURBULENT FLOW

In laminar flow, fluid particles are constrained to motion in parallel paths by the action of viscosity. The shearing stress between adjacent moving layers is determined in laminar flow by the Newton's viscosity law.

$$\tau = \mu\frac{du}{dy} \tag{1.4}$$

The stress is the product of viscosity and velocity gradient (Fig.5.2). If the laminar flow is disturbed by wall roughness or some other obstacle, the disturbances are rapidly damped by viscous action and downstream the flow is smooth again. A laminar flow is stable against such disturbances, but a turbulent flow is not.

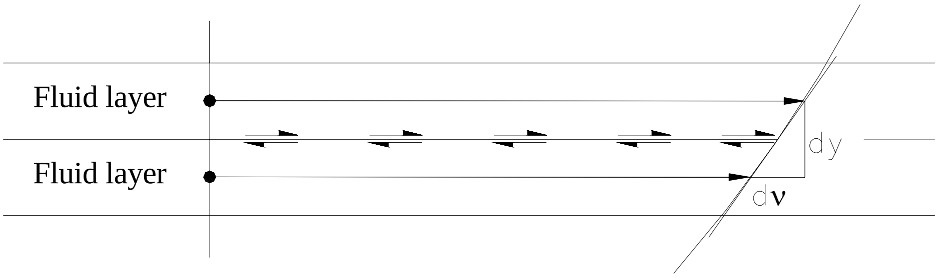


Fig. 5.2

The instability of laminar flow at a high Reynolds number causes disruption of the laminar pattern of fluid motion. With sufficient disturbances the result is known as *turbulence*. Turbulence is characterized by the irregular, chaotic motion of fluid particles. Experiments show that, at any fixed point in a completely developed flow, the *instantaneous velocity* and, consequently, the *instantaneous pressure* fluctuate regularly about a *mean value* with respect to both time and spatial direction. A typical curve showing velocity fluctuations in the x-direction of a turbulent flow is plotted as a function of time in Fig. 5.3.

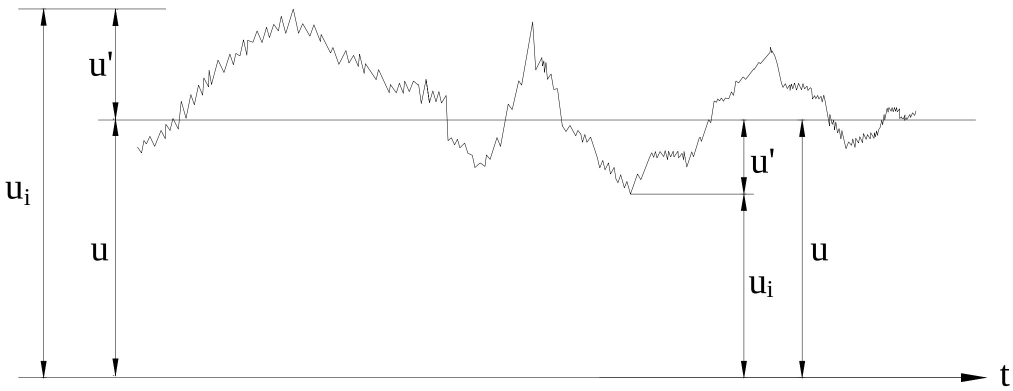


Fig. 5.3

In the theoretical study it is convenient to split the instantaneous velocities, u_i, v_i, and w_i, into their *time average components*, $\bar{u}$, $\bar{v}$, and $\bar{w}$, and *momentary fluctuation components*, u', v', and w'. Thus

$$u_i = \bar{u} + u'$$
$$v_i = \bar{v} + v' \qquad (5.2)$$
$$w_i = \bar{w} + w'$$

and, similarly, for pressure:

$$p_i = \bar{p} + p' \qquad (5.3)$$

In accordance with their definition, the following relationships exit:

$$\frac{1}{T}\int_0^T u_i dt = \overline{u}$$
$$\frac{1}{T}\int_0^T v_i dt = \overline{v} = 0$$
$$\frac{1}{T}\int_0^T w_i dt = \overline{w} = 0 \qquad (5.4)$$
$$\frac{1}{T}\int_0^T p_i dt = \overline{p}$$

Also, the time averages of fluctuation components must be equal to zero:

$$\frac{1}{T}\int_0^T u' dt = \overline{u}' = 0, \overline{v'} = 0, \overline{w'} = 0, \overline{p'} = 0 \qquad (5.5)$$

Let's try to examine the flow in a pipe in (Fig.5.4). dxdz is a cylindrical surface area with the same axis of the pipe. v is the velocity component normal the pipe axis, x. vdxdz is the volume of fluid passing through dxdz in unit time. ρvdxdz is the mass of fluid flowing through this area. (ρvdxdz)u is the momentum of the fluid flowing normal to this area over the x-axis per unit time. Therefore, the momentum increase in the x-axis direction is:

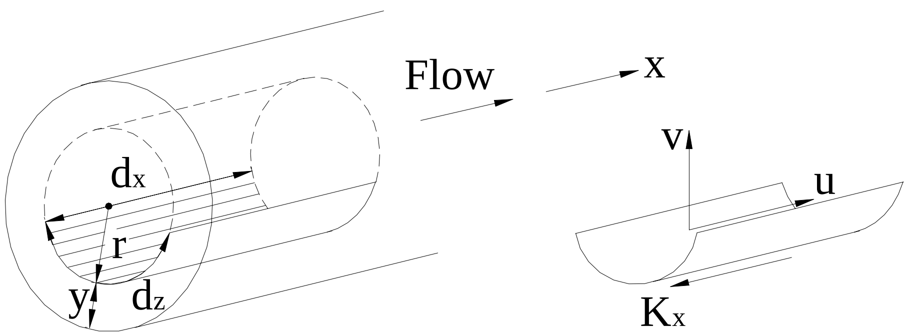


Fig. 5.4

$$\frac{dI_x}{dt} = \rho uvdxdz \qquad (5.6)$$

Here, I_x is the momentum component in the x-axis direction. According to the Newton's second law, the change in the momentum will create a force:

$$-K_x = \frac{dI_x}{dt} \qquad (5.7)$$

K_x is the force created and will act normal to the dxdz surface. Since K_x is taken opposite to the x-axis, there will be a minus sign in front of the force, K_x.

Using Eqs. (5.6) and (5.7),

$$-K_x = \rho uvdxdz \tag{5.8}$$

Shearing stress may be found as for the dxdz area,

$$\frac{K_x}{dxdz} = -\rho uv \tag{5.9}$$

Since u and v changes over their mean value by time, uv term will also change with time. The mean shearing stress may be found by applying Equ. (5.9),

$$\begin{aligned}
\tau &= \lim_{T\to\infty} \frac{1}{T}\int_0^T (-\rho uv)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[-\rho(\bar{u}+u')(0+v')\right]dt \\
&= -\rho\bar{u} \lim_{T\to\infty} \frac{1}{T}\int_0^T v'dt - \rho \lim_{T\to\infty} \frac{1}{T}\int_0^T u'v'dt
\end{aligned}$$

Since the first integral of the last equation is the mean of the fluctuation component, then $\overline{v'} = 0$, and the second integral is the mean value of the product of u′ and v′, $\overline{u'v'}$. Then the mean shearing stress may be found as,

$$\tau = -\rho\overline{u'v'} \tag{5.10}$$

Terms of the form $-\rho\overline{u'v'}$ now called *Reynolds stresses*. In general, the shearing stress of the turbulent flow may be written as in the following form:

$$\tau = \tau_l + \tau_t = \mu\frac{du}{dy} + \left(-\rho\overline{u'v'}\right) \tag{5.11}$$

The first term of Equ. (5.11) is the result of viscous effect and the second term is the result of turbulence effect. If the flow is laminar, u′ and v′ velocity fluctuations will be zero, so Equ. (5.11) will take the form of Equ. (1.4).

In turbulent flow the numerical value of Reynolds stress ($-\rho\overline{u'v'}$) is generally several times greater than that of (μdu/dy). Therefore, the viscosity term (μdu/dy) may be neglected in case of turbulent flow.

5.3.1. Turbulence Viscosity

Shearing stress caused by turbulence effect in Equ. (5.10) can be written in the similar form as the viscous effect shearing stress as

$$-\rho\overline{u'v'} = \mu_T \frac{du}{dy} \tag{5.12}$$

Here μ_T is known as *turbulence viscosity*.

CHAPTER 6

TWO-DIMENSIONAL IDEAL FLOW

6.1 INTRODUCTION

An ideal fluid is purely hypothetical fluid, which is assumed to have no viscosity and no compressibility, also, in the case of liquids, no surface tension and vaporization. The study of flow of such a fluid stems from the eighteenth century hydrodynamics developed by mathematicians, who, by making the above assumptions regarding the fluid, aimed at establishing mathematical models for fluid flows. Although the assumptions of ideal flow appear to be far obtained, the introduction of the boundary layer concept by Prandtl in 1904 enabled the distinction to be made between two regimes of flow: that adjacent to the solid boundary, in which viscosity effects are predominant and, therefore, the ideal flow treatment would be erroneous, and that outside the boundary layer, in which viscosity has negligible effect so that idealized flow conditions may be applied.

The ideal flow theory may also be extended to situations in which fluid viscosity is very small and velocities are high, since they correspond to very high values of Reynolds number, at which flows are independent of viscosity. Thus, it is possible to see ideal flow as that corresponding to an infinitely large Reynolds number and zero viscosity.

6.2. CONTINUITY EQUATION

The control volume ABCDEFGH in Fig. 6.1 is taken in the form of a small prism with sides dx, dy and dz in the x, y and z directions, respectively.

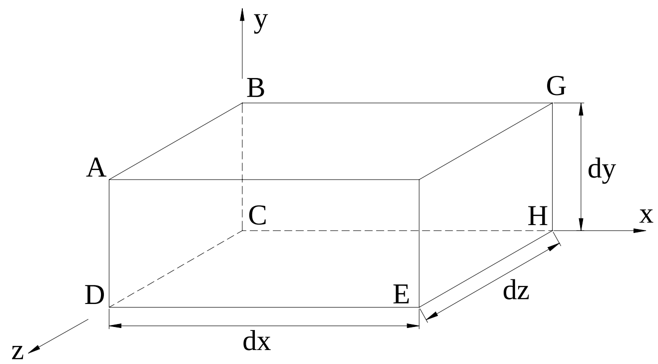


Fig. 6.1

The mean values of the component velocities in these directions are u, v, and w. Considering flow in the x direction,

Mass inflow through ABCD in unit time $= \rho u dy dz$

In the general case, both specific mass ρ and velocity u will change in the x direction and so,

Mass outflow through EFGH in unit time $=\left[\rho u+\frac{\partial(\rho u)}{\partial x}dx\right]dydz$

Thus,

Net outflow in unit time in x direction $=\frac{\partial(\rho u)}{\partial x}dxdydz$

Similarly,

Net outflow in unit time in y direction $=\frac{\partial(\rho v)}{\partial y}dxdydz$

Net outflow in unit time in z direction $=\frac{\partial(\rho w)}{\partial z}dxdydz$

Therefore,

Total net outflow in unit time $=\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]dxdydz$

Also, since ∂ρ/∂t is the change in specific mass per unit time,

Change of mass in control volume in unit time $=-\frac{\partial\rho}{\partial t}dxdydz$

(the negative sign indicating that a net outflow has been assumed). Then,

Total net outflow in unit time = Change of mass in control volume in unit time

$$\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]dxdydz=-\frac{\partial\rho}{\partial t}dxdydz$$

or

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=-\frac{\partial\rho}{\partial t} \tag{6.1}$$

Equ. (6.1) holds for every point in a fluid flow whether steady or unsteady, compressible or incompressible. However, for incompressible flow, the specific mass ρ is constant and the equation simplifies to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{6.2}$$

For two-dimensional incompressible flow this will simplify still further to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6.3}$$

EXAMPLE 6.1: The velocity distribution for the flow of an incompressible fluid is given by u = 3-x, v = 4+2y, w = 2-z. Show that this satisfies the requirements of the continuity equation.

SOLUTION: For three-dimensional flow of an incompressible fluid, the continuity equation simplifies to Equ. (6.2);

$$\frac{\partial u}{\partial x} = -1, \frac{\partial v}{\partial y} = 2, \frac{\partial w}{\partial z} = -1$$

and, hence,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -1 + 2 - 1 = 0$$

Which satisfies the requirement for continuity.

6.3. EULER'S EQUATIONS

Euler's equations for a vertical two-dimensional flow field may be derived by applying Newton's second law to a basic differential system of fluid of dimension dx by dz (Fig. 6.2).

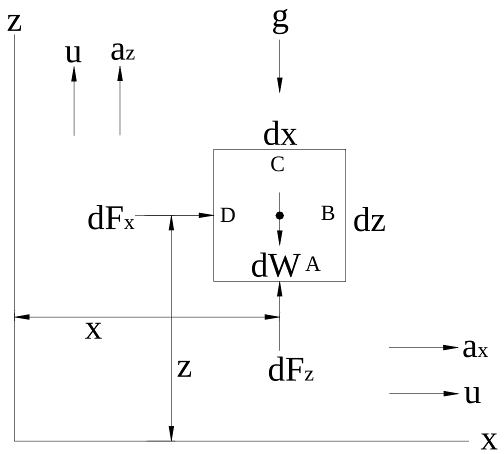


Fig. 6.2

The forces dF_x and dF_z on such an elemental system are,

$$dF_x = -\frac{\partial p}{\partial x} dxdz$$

$$dF_z = -\frac{\partial p}{\partial z} dxdz - \rho g dxdz$$

The accelerations of the system have been derived for steady flow (Equ. 3.5) as,

$$a_x = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}$$

$$a_z = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}$$

Applying Newton's second law by equating the differential forces to the products of the mass of the system and respective accelerations gives,

$$-\frac{\partial p}{\partial x}dxdz = \rho dxdz\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right)$$

$$-\frac{\partial p}{\partial z}dxdz - \rho gdxdz = \rho dxdz\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right)$$

and by cancellation of dxdz and slight arrangement, the *Euler equations* of two-dimensional flow in a vertical plane are

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \tag{6.4}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + g \tag{6.5}$$

Accompanied by the equation continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{6.3}$$

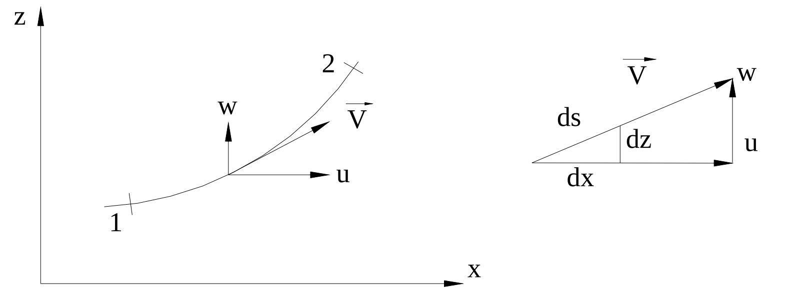
The Euler equations form a set of three simultaneous partial differential equations that are basic to the solution of two-dimensional flow field problems; complete solution of these equations yields p, u and w as functions of x and z, allowing prediction of pressure and velocity at any point in the flow field.

6.4. BERNOULLI'S EQUATION

Bernoulli's equation may be derived by integrating the Euler equations for a constant specific weight flow. Multiplying Equ. (6.4) by dx and Equ. (6.5) by dz and integrating from 1 to 2 on a streamline give

$$\int_1^2 u\frac{\partial u}{\partial x}dx + \int_1^2 w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\int_1^2 \frac{\partial p}{\partial x}dx$$

$$\int_1^2 u\frac{\partial w}{\partial x}dz + \int_1^2 w\frac{\partial w}{\partial z}dz = -\frac{1}{\rho}\int_1^2 \frac{\partial p}{\partial z}dz - g\int_1^2 dz$$



However, along a streamline in any steady flow dz/dx=w/u and therefore udz = wdx. If we collect the both equations,

$$\int_1^2\left(u\frac{\partial u}{\partial x}+w\frac{\partial w}{\partial x}\right)dx+\int_1^2\left(u\frac{\partial u}{\partial z}+w\frac{\partial w}{\partial z}\right)dz=-\frac{1}{\rho}\int_1^2\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial z}dz\right)-g\int_1^2 dz$$

Since $u\dfrac{\partial u}{\partial x}=\dfrac{\partial\left(u^2/2\right)}{\partial x}$, arranging the equation yields,

$$\int_1^2\left[\frac{\partial\left(u^2/2\right)}{\partial x}dx+\frac{\partial\left(u^2/2\right)}{\partial z}dz\right]+\int_1^2\left[\frac{\partial\left(w^2/2\right)}{\partial x}dx+\frac{\partial\left(w^2/2\right)}{\partial z}dz\right]=-\frac{1}{\rho}\int_1^2\left[\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial z}dz\right]-g\int_1^2 dz$$

Since the terms in each bracket is a total differential, by integrating gives

$$\frac{u_2^2}{2}-\frac{u_1^2}{2}+\frac{w_2^2}{2}-\frac{w_1^2}{2}=-\frac{1}{\rho}\left(p_2-p_1\right)-g\left(z_2-z_1\right)$$

By remembering that $V^2 = u^2 + w^2$, the equation takes the form of

$$\frac{V_1^2}{2g}+\frac{p_1}{\gamma}+z_1=\frac{V_2^2}{2g}+\frac{p_2}{\gamma}+z_2 \qquad (6.6)$$

This equation is the well-known Bernoulli equation and valid on the streamline between points 1 and 2 in a flow field.

6.5. ROTATIONAL AND IRROTATIONAL FLOW

Considerations of ideal flow lead to yet another flow classification, namely the distinction between rotational and irrotational flow.

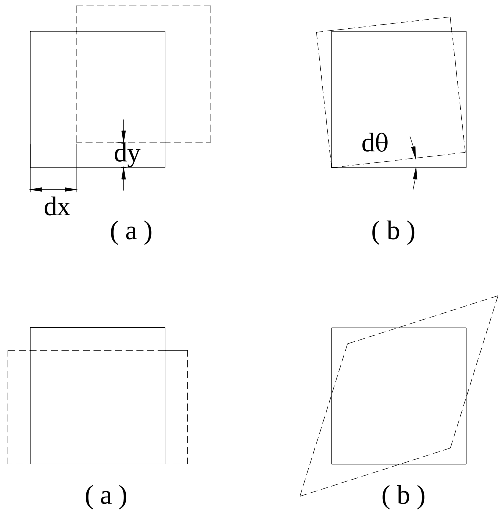


Fig. 6.2 and 6.3

Basically, there are two types of motion: translation and rotation. The two may exist independently or simultaneously, in which case they may be considered as one superimposed on the other. If a solid body is represented by square, then pure translation or pure rotation may be represented as shown in Fig. 6.2 (a) and (b), respectively.

If we now consider the square to represent a fluid element, it may be subjected to deformation. This can be either linear or angular, as shown in Fig. 6.3 (a) and (b), respectively.

The rotational movement can be specified in mathematical terms. Fig.6.4 shows the rotation of a rectangular fluid element in a two-dimensional flow.

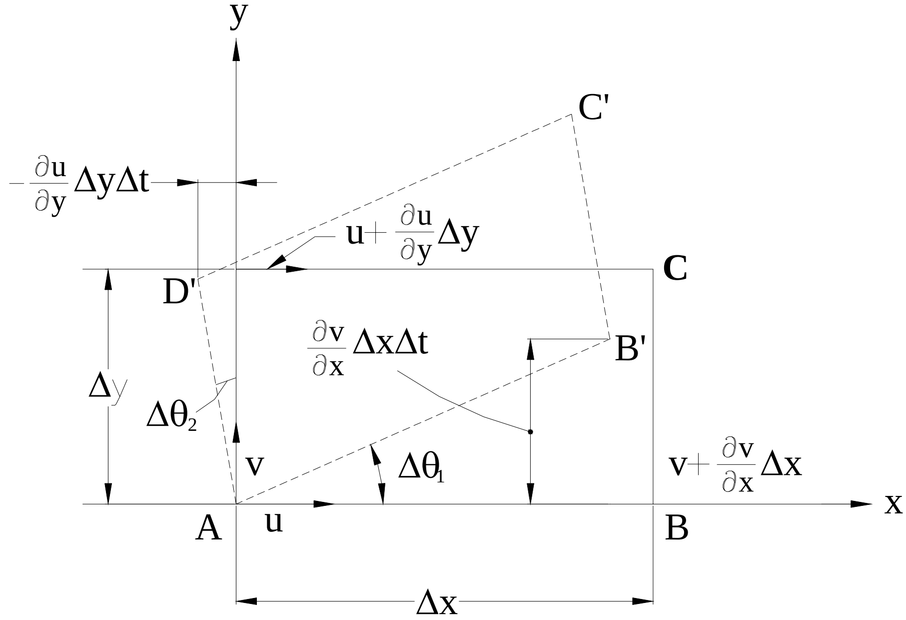


Fig. 6.4

During the time interval Δt the element ABCD has moved relative to A to a new position, which is indicated by the dotted lines. The angular velocity (w_{AB}) of line AB is,

$$w_{AB} = \lim_{\Delta t \to 0} \frac{\Delta \theta_1}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\partial v / \partial x)\Delta x \Delta t}{\Delta x \Delta t} = \frac{\partial v}{\partial x}$$

Similarly, the angular velocity (w_{AD}) of line AD is

$$w_{AD} = \lim_{\Delta t \to 0} \frac{\Delta \theta_2}{\Delta t} = -\frac{\partial u}{\partial y}$$

The average of the angular velocities of these two line elements is defined as the *rotation* w of the fluid element ABCD. Therefore,

$$w = \frac{1}{2}\left(w_{AB} + w_{AD}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{6.7}$$

The condition of irrotationality for a two-dimensional flow is satisfied when the rotation w is everywhere zero, so that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{6.8}$$

For a three-dimensional flow, the condition of irrotationality requires that the rotation about each of three axes, which are parallel to x, y and z-axes must be zero. Therefore, the following three equations must be satisfied:

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{6.9}$$

EXAMPLE 6.2: The velocity components in a two-dimensional velocity field for an incompressible fluid are expressed as

$$u = \frac{y^3}{3} + 2x - x^2 y$$

$$v = xy^2 - 2y - \frac{x^3}{3}$$

Show that these functions represent a possible case of an irrotational flow.

SOLUTION: The functions given satisfy the continuity equation (Equ. 6.3), for their partial derivatives are

$$\frac{\partial u}{\partial x} = 2 - 2xy \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 2xy - 2$$

so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0$$

Therefore they represent a possible case of fluid flow. The rotation w of any fluid element in the flow field is,

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\left(xy^2 - 2y - \frac{x^3}{3}\right) - \frac{\partial}{\partial y}\left(\frac{y^3}{3} + 2x - x^2 y\right)\right]$$

$$= \frac{1}{2}\left[\left(y^2 - x^2\right) - \left(y^2 - x^2\right)\right] = 0$$

6.6. CIRCULATION AND VORTICITY

Consider a fluid element ABCD in rotational motion. Let the velocity components along the sides of the element be as shown in Fig. 6.5.

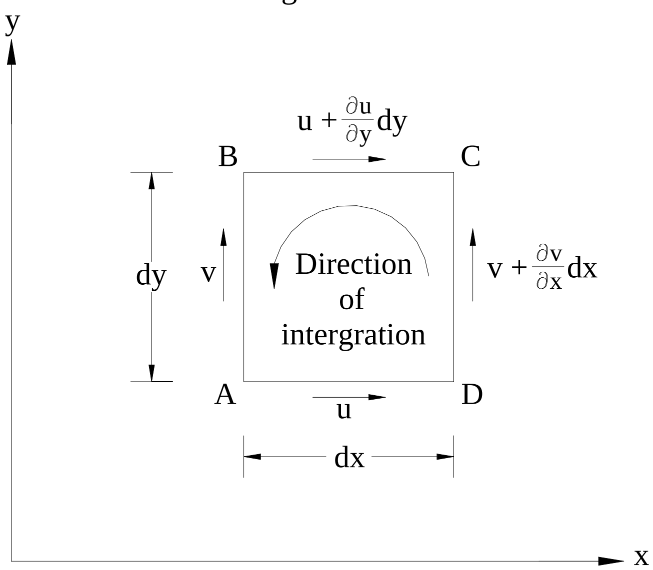


Fig. 6.5

Since the element is rotating, being part of rotational flow, there must be a resultant peripheral velocity. However, since the center of rotation is not known, it is more convenient to relate rotation to the sum of products of velocity and distance around the contour of the element. Such a sum is the line integral of velocity around the element and it is called *circulation*, denoted by Γ. Thus,

$$\Gamma = \oint \vec{V} \cdot d\vec{s} \tag{6.10}$$

Circulation is, by convention, regarded as positive for anticlockwise direction of integration. Thus, for the element ABCD, from side AD

$$\Gamma_{ABCD} = udx + \left(v + \frac{\partial v}{\partial x} dx \right) dy - \left(u + \frac{\partial u}{\partial y} dy \right) dx - vdy$$

$$= \frac{\partial v}{\partial x} dxdy - \frac{\partial u}{\partial y} dydx$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dydx$$

Since

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \zeta$$

For the two-dimensional flow in the x-y Ë–aWe, is the *vorticity* of the element about the z-axis, ζ_z. ThÔ product dxdy is the area of the element dA.

Th)s

$$\Gamma_{ABCD} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = \zeta_z dA$$

It is seen, therefore, that the circulation[XrWund X contour is equal to the sum of the vorticities within%the area of contour. This is known as Stokes' thXo em and may be stated mathematically, for a general case of any contoué›C (Fig_6.6) as

$$\Gamma_C = \oint V_s \cdot ds = \int_A \zeta \cdot dA \qquad (6.11)$$

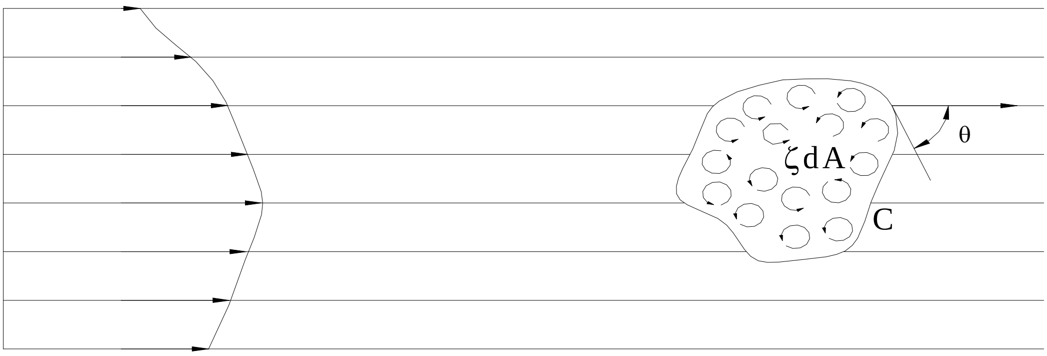


Fig. 6.6

The above considerations<indicate that, for irrotational flow, since vorticity iÏ equal to zero, Óhe circulation around a closed contour through which fluid és moving, must be equal to zero.

6.7. STREAM FUNCTIO…

A *stream function* ψ is a mathematical device, which describes the formŒof any particular pattern of flow. In Fig. 6.7 let P (x, y) represent a movable point in the plane of

motion of a steady, two-dimensional flow, and cönsider the flow to have unit thicrness perpendicular to the xy-plane.

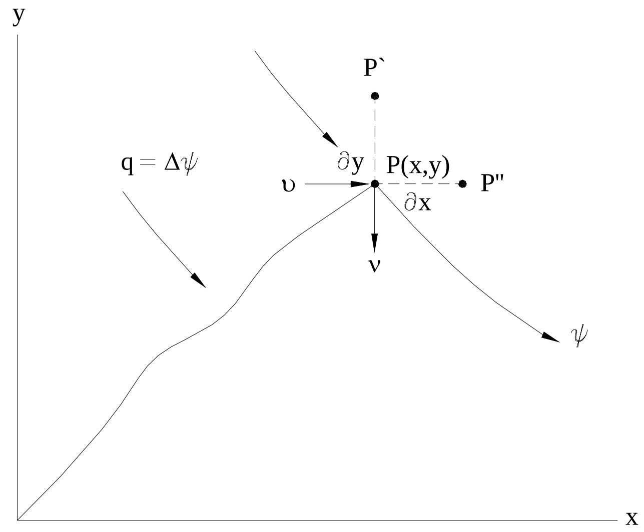


Fig. 6.7

The volume rate of flow across any line connecting OP is a function of the position of P and defined as the stream function ψ:

$$\psi = f(x, y)$$

Stream function ψ has a unit of cubic meter per second per meter thickness (normal to the xy-plane).

The two components of velocity, u and v can be expressed in terms of ψ. If the point P in Fig. 6.7 is displaced an infinitesimal distance ∂y is ∂ψ = u·∂y. Therefore,

$$u = \frac{\partial \psi}{\partial y} \qquad (6.12)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} \qquad (6.13)$$

When these values of u and v are substituted into Eqs. (3.6), the differential equation for streamlines in two-dimensional flow becomes

$$\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = 0$$

By definition, the left-hand side of this equation is equal to the total differential dψ when ψ = f (x, y). Thus,

$$d\psi = 0$$

and

$$\psi = C \ \text{(constant along a streamline)} \qquad (6.14)$$

Equ. (6.14) indicates that the general equation for the streamlines in a flow pattern is obtained when ψ is equated to a constant. Different numerical values of the constant in turn define streamlines. As an example, the stream function for a steady two-dimensional flow at 90^0 corner (shown in Fig. 6.8) takes the following form:

$$\psi = xy$$

The general equation for the streamlines of such a flow is obtained when ψ = C (constant), that is,

$$xy = C$$

Which indicates that the streamlines are a family of rectangular hyperbolas. Different numerical values of C define different streamlines as shown in Fig. 6.8. Obviously, the volume rate of flow between any two streamlines is equal to the difference in numerical values of their constants.

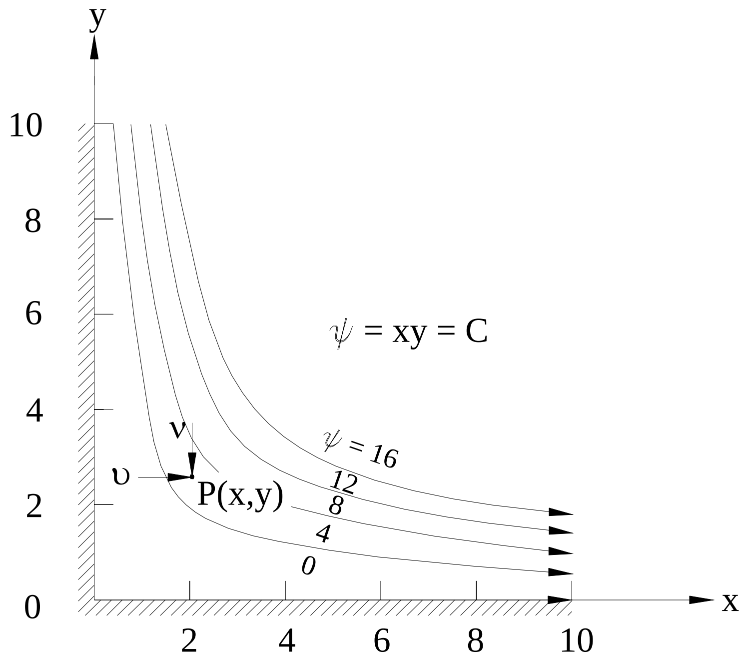


Fig. 6.8

EXAMPLE 6.3: A stream function is given by

$$\psi = 3x^2 - y^3$$

Determine the magnitude of velocity components at the point (3,1).

SOLUTION: The x and y components of velocity are given by

x-component: $u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 - y^3\right) = -3y^2$

y-component: $v=-\frac{\partial \psi}{\partial x}=-\frac{\partial}{\partial x}\left(3x^2-y^3\right)=-6x$

At the point (3,1)

u = -3 and v = -18

and the total velocity is the vector sum of the two components.

$$\vec{V}=-3\vec{i}-18\vec{j}$$

Note that $\partial u/\partial x=0$ and $\partial v/\partial y=0$, so that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Therefore the given stream function satisfies the continuity equation.

The equation for vorticity,

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{6.14}$$

may also be expressed in terms of ψ by substituting Eqs. (6.12) and (6.13) into Equ. (6.14)

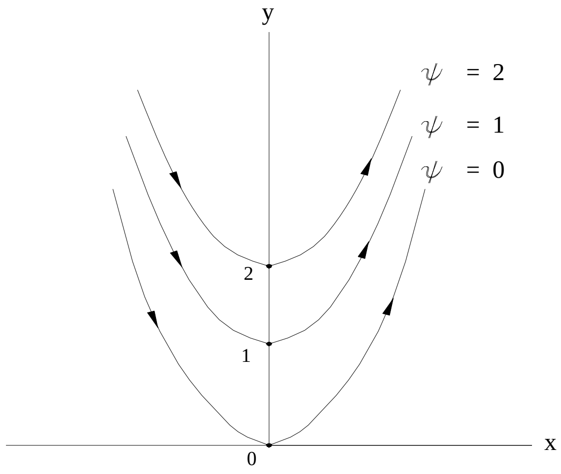
$$\zeta=-\frac{\partial^2 \psi}{\partial x^2}-\frac{\partial^2 \psi}{\partial y^2}$$

However, for irrotational flows, ζ = 0, and the classic Laplace equation,

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=\nabla^2\psi=0$$

results. This means that the stream functions of all irrotational flows must satisfy the Laplace equation and that such flows may be identified in this manner; conversely, flows whose ψ does not satisfy the Laplace equation are rotational ones. Since both rotational and irrotational flow fields are physically possible, the satisfaction of the Laplace equation is no criterion of the physical existence of a flow field.

EXAMPLE 6.4: A flow field is described by the equation $\psi = y-x^2$. Sketch the streamlines ψ = 0, ψ=1, and ψ = 2. Derive an expression for the velocity V at any point in the flow field. Calculate the vorticity.



SOLUTION: From the equation for ψ, the flow field is a family of parabolas symmetrical about the y-axis with the streamline ψ = 0 passing through the origin of coordinates.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(y - x^2\right) = 1$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(y - x^2\right) = 2x$$

Which allows the directional arrows to be placed on streamlines as shown.
The magnitude V of the velocity may be calculated from

$$V = \sqrt{u^2 + v^2} = \sqrt{1 + 4x^2}$$

and the vorticity by Equ. (6.14)

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(1) = 2\sec^{-1} \quad \text{(Counter clockwise)}$$

Since $\zeta \neq 0$, this flow field is seen to be rotational one.

6.8. VELOCITY POTENTIAL FUNCTIONS

When the flow is irrotational, a mathematical function called the *velocity potential* function ϕ may also be found to exist. A velocity potential function ϕ for a steady, irrotational flow in the xy-plane is defined as a function of x and y, such that the partial derivative ϕ with respect to displacement in any chosen direction is equal to the velocity in that direction. Therefore, for the x and y directions,

$$u = \frac{\partial \phi}{\partial x} \tag{6.15}$$

$$v = \frac{\partial \phi}{\partial y} \tag{6.16}$$

These equations indicate that the velocity potential increases in the direction of flow. When the velocity potential function ϕ is equated to a series of constants, equations for a family of equipotential lines are the result.

The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.3}$$

may be written in terms of ϕ by substitution Eqs. (6.15) and (6.16) into the Equ. (6.3), to yield The Laplacian differential equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0 \tag{6.17}$$

Thus all practical flows (which must conform to the continuity principle) must satisfy the Laplacian equation in terms of ϕ.

Similarly, the equation of vorticity,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{6.14}$$

may be put in terms of ϕ to give

$$\zeta = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}$$

from which a valuable conclusion may be drawn: Since,

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial x \partial y}$$

the vorticity must be zero for the existence of a velocity potential. From this it may be deduced that only irrotational ($\zeta = 0$) flow fields can be characterized by a velocity potential ϕ; for this reason *irrotational* flows are also known as *potential* flows.

RELATION BETWEEN STREAM FUNCTION AND VELOCITY POTENTIAL

A geometric relationship between streamlines and equipotential lines may be derived from the foregoing equations and restatement of certain mathematical definitions; the latter are (with definitions of u and v inserted)

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = udx + vdy$$

However, along a streamline ψ is constant and $d\psi = 0$, so along a streamline,

$$\frac{dy}{dx} = \frac{v}{u}$$

also along any equipotential line ϕ is constant and $d\phi = 0$, so along an equipotential line;

$$\frac{dy}{dx} = -\frac{v}{u}$$

The geometric significance of this is seen in Fig. 6.9. *The equipotential lines are normal to the streamlines*. Thus the streamlines and equipotential lines (for potential flows) form a net, called a *flow net*, of mutually perpendicular families of lines, a fact of great significance for the study of flow fields where formal mathematical expressions of ϕ and ψ are unobtainable. Another feature of the velocity potential is that the value of ϕ drops *along the direction of the flow*, that is, $\phi_3<\phi_2<\phi_1$.

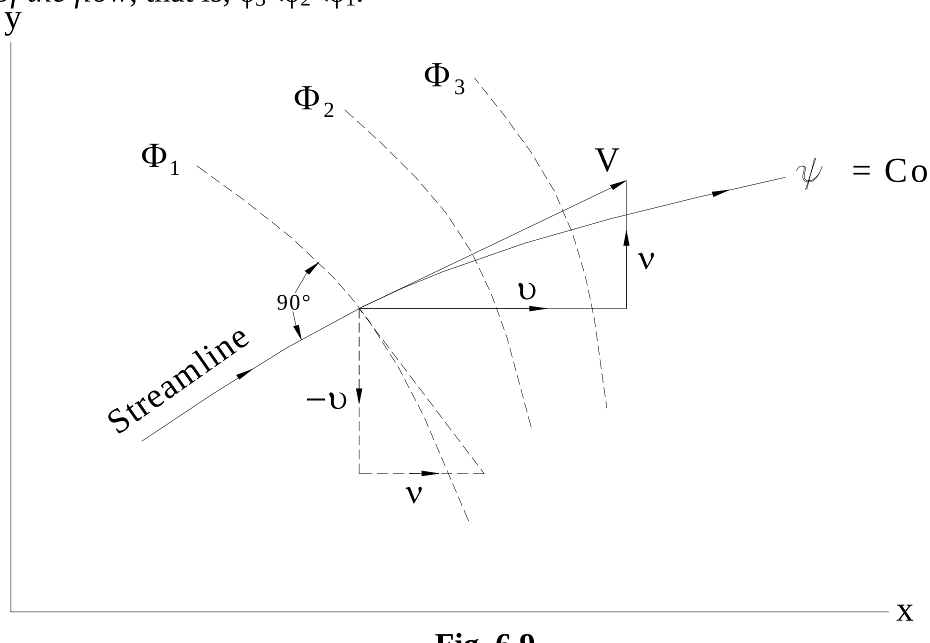


Fig. 6.9

It is important to note that the stream functions are not restricted to irrotational (potential) flows, whereas the velocity potential function exists only when the flow is irrotational because the velocity potential function always satisfies the condition of irrotationality (Equ. 6.8). The partial derivative of u in Equ. (6.15) is always equal to the partial derivative v in Equ. (6.16)

For any flow pattern the velocity potential function ϕ is related to the stream function ψ by the means of the two velocity components, u and v, at any point (x, y) in the Cartesian coordinate system in the form of the two following equations:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{6.18}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{6.19}$$

EXAMPLE 6.5: A stream function in a two-dimensional flow is ψ = 2xy. Show that the flow is irrotational (potential) and determine the corresponding velocity potential function ϕ.

SOLUTION: The given stream function satisfies the condition of irrotationality, that is,

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}(2xy) + \frac{\partial^2}{\partial y^2}(2xy)\right] = 0$$

which shows that the flow is irrotational. Therefore, a velocity potential function ϕ will exist for this flow.

By using Equ. (6.18)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

Therefore,

$$\phi = \int 2x \partial x = x^2 + f_1(y) \qquad \text{(a)}$$

From Equ. (6.19)

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(2xy) = -2y$$

From this equation,

$$\phi = \int -2y \partial y = -y^2 + f_2(x) \qquad \text{(b)}$$

The velocity potential function,

$$\phi = x^2 - y^2 + C$$

satisfies both ϕ functions in Equations a and b.

EXAMPLE 6.6: In a two-dimensional, incompressible flow the fluid velocity components are given by: u = x – 4y and v = -y - 4x. Show that the flow satisfies the continuity equation and obtain the expression for the stream function. If the flow is potential (irrotational) obtain also the expression for the velocity potential.

SOLUTION: For incompressible, two-dimensional flow, the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

but u = x – 4y and v = -y – 4x.

$$\frac{\partial u}{\partial x}=1 \qquad \frac{\partial v}{\partial y}=-1$$

Therefore, 1 – 1 = 0 and the flow satisfies the continuity equation.

To obtain the stream function, using Eqs. (6.12) and (6.13)

$$u=\frac{\partial \psi}{\partial y}=x-4y \qquad \text{(a)}$$

$$v=-\frac{\partial \psi}{\partial x}=y+4x \qquad \text{(b)}$$

Therefore, from (a),

$$\psi=\int (x-4y)\partial y+f(x)+C$$

$$=xy-2y^2+f(x)+C$$

But, if $\psi_0 = 0$ at x = 0 and y = 0, which means that the reference streamline passes through the origin, then C = 0 and

$$\psi=xy-2y^2+f(x) \qquad \text{(c)}$$

To determine f (x), differentiate partially the above expression with respect to x and equate to –v, equation (b):

$$\frac{\partial \psi}{\partial x}=y+\frac{\partial}{\partial x}f(x)=y+4x$$

$$f(x)=\int 4x\partial x=2x^2$$

Substitute into (c)

$$\psi = 2x^2 + xy - 2y^2$$

To check whether the flow is potential, there are two possible approaches:

(a) Since

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

but

$$v = -(4x + y) \qquad \text{and} \qquad u = x - 4y$$

Therefore,

$$\frac{\partial v}{\partial x} = -4 \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -4$$

so that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -4 + 4 = 0$$

and flow is potential.

(a) Laplace's equation must be satisfied,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

$$\psi = 2x^2 + xy - 2y^2$$

Therefore,

$$\frac{\partial \psi}{\partial x} = 4x + y \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = x - 4y$$

$$\frac{\partial^2 \psi}{\partial x^2} = 4 \qquad \text{and} \qquad \frac{\partial^2 \psi}{\partial y^2} = -4$$

Therefore 4 – 4 = 0 and flow is potential.

Now, to obtain the velocity potential,

$$\frac{\partial \phi}{\partial x} = u = x - 4y$$

$$\phi = \int (x - 4y)\partial x + f(y) + G$$

But $\phi_0 = 0$ at x = 0 and y = 0, so that G = 0. Therefore,

$$\phi = \frac{x^2}{2} - 4yx + f(y)$$

Differentiating with respect to y and equating to v,

$$\frac{\partial \phi}{\partial y} = -4x + \frac{d}{dy} f(y) = -4x - y$$

$$\frac{d}{dy} f(y) = -y \qquad \text{and} \qquad f(y) = -\frac{y^2}{2}$$

so that

$$\phi = \frac{x^2}{2} - 4yx - \frac{y^2}{2}$$

6.10. THE FLOW NET

In any two-dimensional steady flow problem, the mathematical solution is to determine the velocity field of flow expressed by the following two velocity components:

$$u = f_1(x, y)$$
$$v = f_2(x, y)$$

However, if the flow is irrotational, the problem can also be solved graphically by means of a *flow net* such as the one shown in Fig.6.10. This is a network of mutually perpendicular streamlines and equipotential lines. The streamlines, which show the direction of flow at any point, are so spaced that there is an equal rate of flow Δq discharging through each stream tube. The discharge Δq is equal to the change in ψ from one streamline to the next. The equipotential lines are then drawn everywhere normal to the streamlines. The spacings of equipotential lines are selected in such a way that the change in velocity potential from one equipotential line to the next is constant. Furthermore, that is, $\Delta\psi = \Delta\phi$. As a result they form approximate squares (Fig. 6.10)

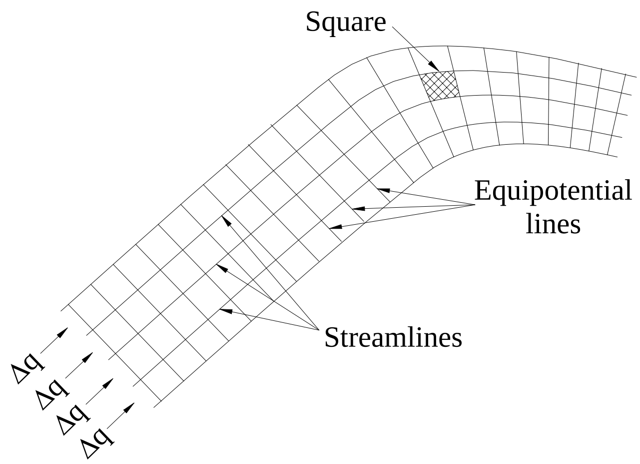


Fig.6.10

From the continuity relationship, the distances between both sets of lines must therefore be inversely proportional to the local velocities. Thus the following relation is a key to the proper construction of any flow net.

$$\frac{v_1}{v_2} = \frac{\Delta n_2}{\Delta n_1} = \frac{\Delta s_2}{\Delta s_1}$$

Where Δn and Δs are respectively the distance between streamlines and between equipotential lines.

Since there is only one possible pattern of flow for a given set of boundary conditions, a flow net, if properly constructed, represents a unique mathematical solution for a steady, irrotational flow. Whenever the flow net is used, the hydrodynamic condition of irrotationality (Equ. 6.8) must be satisfied.

The flow net must be used with caution. The validity of the interpretation depends on the extent to which the assumption of ideal (nonviscous) fluid is justified. Fortunately, such fluids as water and air have rather small viscosity so that, under favorable conditions of flow, the condition of irrotationality can be approximately attained. In practice, flow nets can be constructed for both the flow within solid boundaries (Fig.6.11) and flow around a solid body (Fig. 6.12).

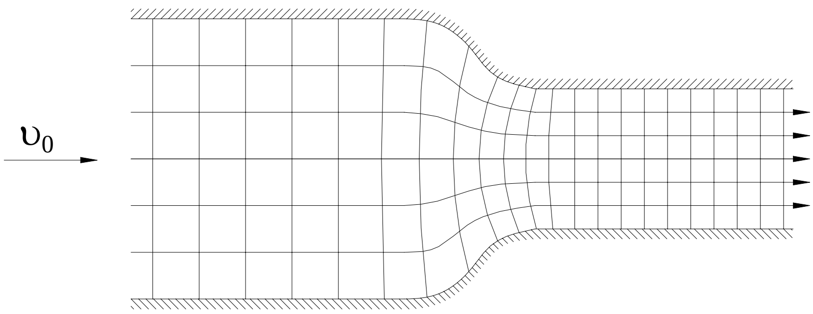


Fig. 6.11

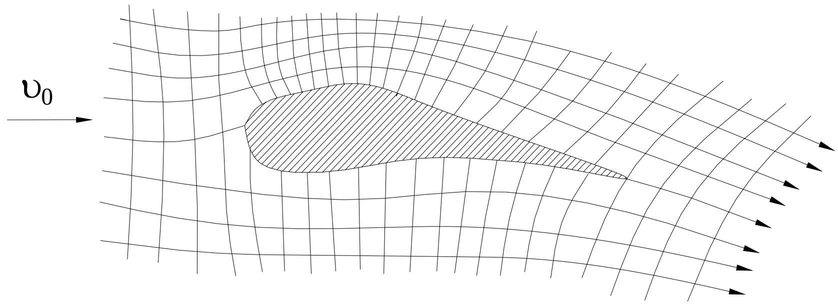


Fig. 6.12

In either flow the boundary surfaces also represent streamlines. Other streamlines are then sketched in by eye. Next, the equipotential lines are drawn everywhere normal to the streamlines. The accuracy of the flow net depends on the criterion that both sets of lines must form approximate squares. Usually a few trials will be required before a satisfactory flow net is produced.

After a correct flow net is obtained, the velocity at any point in the entire field of flow can be determined by measuring the distance between the streamlines (or the equipotential lines), provided the magnitude of velocity at a reference section, such as the velocity of flow v_0 in the straight reach of the channel in Fig. 6.11, or the velocity of approach v_0 in Fig. 6.12, is known. It is seen from both that the magnitude of local velocities depends on the configuration of the boundary surface. Both flow nets give an accurate picture of velocity distribution in the entire field of flow, except for those regions in the vicinity of solid boundaries where the effect of fluid viscosity becomes appreciable.

6.11. GROUND WATER FLOW

The flow net and flow field superposition techniques may also be applied to the flow of real fluids under some restrictions, which are frequently encountered in engineering practice. Consider the one-dimensional flow of an incompressible real fluid in a stream tube. The Bernoulli equation written in differential form is

$$d\left(\frac{p}{\gamma}+\frac{V^2}{2g}+z\right)=-dh_L$$

Suppose now that V is small (so that $dV^2/2g$ may be neglected) and the head loss dh_L given by

$$dh_L=\frac{1}{K}Vdl \tag{6.20}$$

in which dl is the differential length along the stream tube and K is a constant. The Bernoulli equation above then reduces to

$$d\left(\frac{p}{\gamma}+z\right)=-\frac{1}{K}Vdl$$

$$V=-\frac{d}{dl}K\left(\frac{p}{\gamma}+z\right)$$

and, if this may be extended to the two-dimensional case,

$$u=-\frac{\partial}{\partial x}K\left(\frac{p}{\gamma}+z\right)=\frac{\partial\phi}{\partial x} \qquad (6.21)$$

$$v=-\frac{\partial}{\partial x}K\left(\frac{p}{\gamma}+z\right)=\frac{\partial\phi}{\partial y} \qquad (6.22)$$

and $K\left(\frac{p}{\gamma}+z\right)$ is seen to be the velocity potential of such flow field.

The conditions of the foregoing hypothetical problem are satisfied when fluid flows in a laminar condition through a homogenous porous medium. The media interest are those having a set of interconnected pores that will pass a significant volume of fluid, for example, sand, and the certain rock formations. The head-loss law (Equ. 6.20) is usually written as

$$V=K\frac{dh_L}{dl}=-K\frac{dh}{dl}$$

(where $h=p/\gamma+z$) and is an experimental relation called Darcy's law; K is known as the *coefficient of permeability*, has the dimensions of velocity, and ranges in value from 3×10^{-11} m/sec for clay to 0.3 m/sec for gravel.

A Reynolds number is defined for porous media flow as Re = Vd/ν, where V is the apparent velocity or specific discharge (Q/A) and d is a characteristic length of the medium, for example, the effective or median grain size in sand. When Re<1 the flow is laminar and Darcy's linear law is valid. If Re>1 it is likely that the flow is turbulent, that $V^2/2g$ is not negligible, and Equ. 6.20 is not valid. Note that V is not the actual velocity in the pores, but is the velocity obtained by measuring the discharge Q through an area A. The average velocity in the pores is $V_p = V/n$ where n is the porosity of the medium;

n=(Volume of voids) / (Volume of solids plus voids)

Even though the actual fluid flow in the porous medium is viscous dominated and rotational, the "apparent flow" represented by V and the velocities u and v (Equations 6.21 and 6.22) is irrotational. Both the flow net and superposition of flow field concepts can be used. The flow net is very useful in obtaining engineering information for the "seepage flow" of water through or under structures, to wells and under drains, or for the flow of petroleum

through the porous materials of subsurface “reservoirs”. Flow field superposition is most useful in defining the flow pattern in ground water aquifers under the action of recharge and withdrawals wells.

CHAPTER 7

TWO-DIMENSIONAL FLOW OF THE REAL FLUIDS

7.1. INTRODUCTION

Two-dimensional flow problems may easily be solved by potential flow approach as was explained in Chapter 6.

In order to use the ideal fluid assumption for the flow of real fluids, shearing stress that occurs during the fluid motion should be so small to affect the motion. Since shearing stress may be calculated by Newton's viscosity law by $\tau = \mu du/dy$, two conditions should be supplied to have small shearing stresses as;

a) *The viscosity of the fluid must be small*: the fluids as water, air, and etc can supply this condition. This assumption is not valid for oils.
b) *Velocity gradient must be small*: This assumption cannot be easily supplied because the velocity of the layer adjacent to the surface is zero. In visualizing the flow over a boundary surface it is well to imagine a very thin layer of fluid adhering to the surface with a continuous increase of velocity of the fluid. This layer is called as *viscous sublayer*.

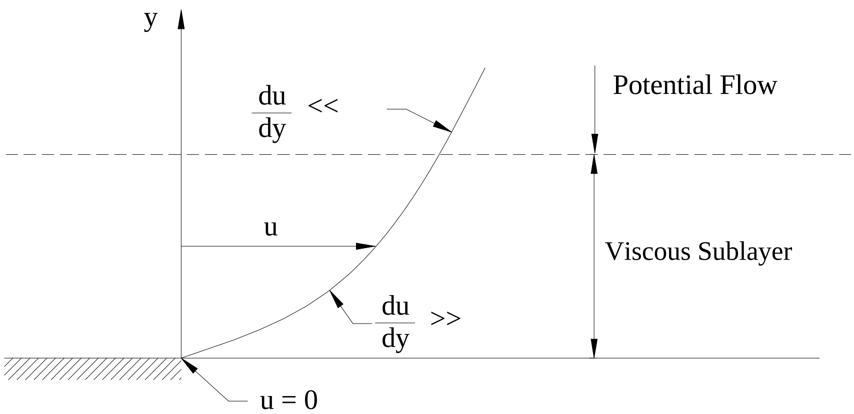


Fig. 7.1

Flow field may be examined by dividing to two zones.

a) *Viscous sublayer zone*: In this layer, velocity gradient is high and the flow is under the affect of shearing stress. Flow motion in this zone must be examined as real fluid flow.
b) *Potential flow zone*: The flow motion in this zone may be examined as ideal fluid flow (potential flow) since velocity gradient is small in this zone.

7.2. BASIC EQUATIONS

Continuity equation for two-dimensional real fluids is the same obtained for two-dimensional ideal fluid. (Equ. 6.3)

Head (energy) loss h_L must be taken under consideration in the application of energy equation. Hence, the Bernoulli equation (Equ. 6.6) may be written as,

$$\frac{V_1^2}{2g}+\frac{p_1}{\gamma}+z_1=\frac{V_2^2}{2g}+\frac{p_2}{\gamma}+z_2+h_L \qquad (7.1)$$

On the same streamline between points 1 and 2 in a flow field.

Forces arising from the shearing stress must be added to the Euler's equations (Eqs. 6.4 and 6.5) obtained for two-dimensional ideal fluids.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (7.2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-g+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad (7.3)$$

These equations are called *Navier-Stokes* equations. The last terms in the parentheses on the right side of the equations are the result of the viscosity effect of the real fluids. If $\upsilon \to 0$, the Navier-Stokes equations take the form of Euler equations. (Eqs. 6.4 and 6.5)

7.3.TWO-DIMENSIONAL LAMINAR FLOW BETWEEN TWO PARALLEL FLAT PLANES

Continuity equation for two-dimensional flow,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Since the flow is uniform, $\partial u/\partial x = 0$, therefore $\partial v/\partial y = 0$. (Fig. 7.2). Using the boundary conditions, for y = 0, u = 0 and $v \cong 0$.

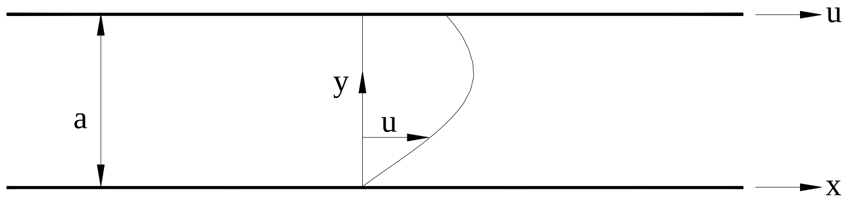


Fig. 7.2

Writing the Navier-Stokes equation for the x-axis,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\vartheta\frac{\partial^2 u}{\partial x^2}+\vartheta\frac{\partial u^2}{\partial y^2}$$

yields

$$\vartheta\frac{d^2u}{dy^2}=\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Since υ = μ/ρ, by taking integration

$$u=\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2}+C_1y+C_2$$

C_1 and C_2 integration constants may be found by using boundary conditions. For y = 0, u = 0, and y = a, u = U.

The velocity distribution equation over the y-axis may be found as,

$$u=U\frac{y}{a}-\frac{ay}{2\mu}\frac{\partial p}{\partial x}\left(1-\frac{y}{a}\right) \tag{7.4}$$

$\frac{\partial p}{\partial x}<0$ in the flow direction.

If we write Navier-Stokes equation for the y-axis, and by using the above defined conditions,

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-g+\vartheta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

yields

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=-g \quad \text{and} \quad \frac{\partial p}{\partial y}=-\gamma$$

This is the hydrostatic pressure distribution that was obtained before.

Special cases;

a) If $\frac{\partial p}{\partial x}=0$, then

$$u=U\frac{y}{a} \tag{7.5}$$

This flow is known as *Coutte flow*. The velocity distribution is linear across the channel section. (Fig. 7.3)

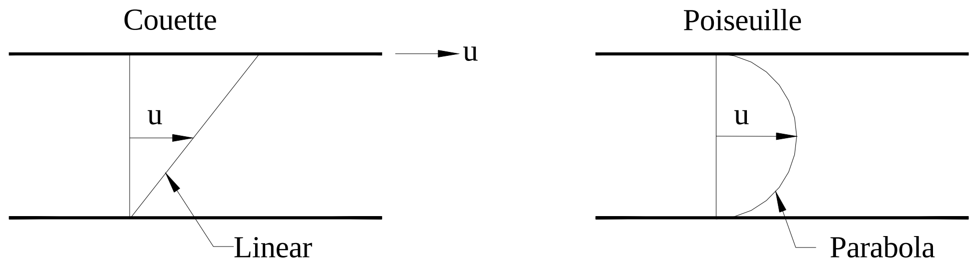


Fig. 7.3

b) If the upper plate is stationary, u = 0. The velocity distribution will take the form of

$$u = -\frac{ay}{2\mu}\frac{\partial p}{\partial x}\left(1 - \frac{y}{a}\right) \tag{7.6}$$

The velocity distribution curve of the laminar is a parabola with its vertex at the centerline of the flow channel. The pressure gradient is negative since there is a pressure drop in flow direction. The maximum velocity occurs at the center of the channel for y = a/2, that is

$$u_{\max} = -\frac{a^2}{8\mu}\frac{\partial p}{\partial x} \tag{7.7}$$

Since dq = udy, the discharge q of the laminar flow per unit width of the channel may be found by the integration of dq. Thus,

$$\begin{aligned} q &= \int_0^a \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(ay - y^2\right)dy \\ &= \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right)a^3 \end{aligned} \tag{7.8}$$

The average velocity V is,

$$V = \frac{q}{b} = \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right)a^2 \tag{7.9}$$

Which is two-thirds of the maximum velocity, u_{max}.

The pressure drop ($p_1 - p_2$) between any two chosen sections 1 and 2 in the direction of flow at a distance L = ($x_2 - x_1$) apart can be determined by integrating (Equ.7.9) with respect to x since the flow is steady;

$$\int_{p_1}^{p_2} -\partial p = \int_{x_1}^{x_2} \frac{12\mu V}{a^2}\partial x$$

and

$$p_1 - p_2 = \frac{12\mu V}{a^2}\left(x_2 - x_1\right) = \frac{12\mu V}{a^2}L \tag{7.10}$$

When the channel is inclined, the term ($-\partial p/\partial x$) in these equations is replaced by $-\partial(p+\gamma z)/\partial x$, ant the term p_1-p_2 on the left-hand side of Equ. (7.10) then becomes $(\gamma z_1+p_1)-(\gamma z_2+p_2)$.

7.4. LAMINAR FLOW IN CIRCULAR PIPES: HAGEN-POISEUILLE THEORY

The derivation of the Hagen-Poiseuille equation for laminar flow in straight, circular pipes is based on the following two assumptions;

a) The viscous property of fluid follows Newton's law of viscosity, that is, $\tau=\mu(du/dy)$,
b) There is no relative motion between fluid particles and solid boundaries, that is, no slip of fluid particles at the solid boundary.

Fig. 7.4 illustrates the laminar motion of fluid in a horizontal circular pipe located at a sufficiently great distance from the entrance section when a steady laminar flow occurs in a straight stretch of horizontal pipe, a pressure gradient must be maintained in the direction of flow to overcome the frictional forces on the concentric cylindrical surfaces, as shown in Fig. 7.4.

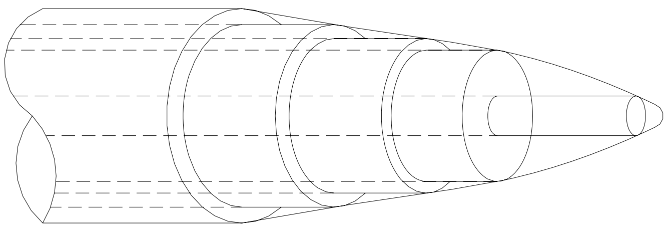


Fig. 7.4

Each concentric cylindrical layer of fluid is assumed to slide over the other in an axial direction. For practical purposes the pressure may be regarded as distributed uniformly over any chosen cross section of the pipe.

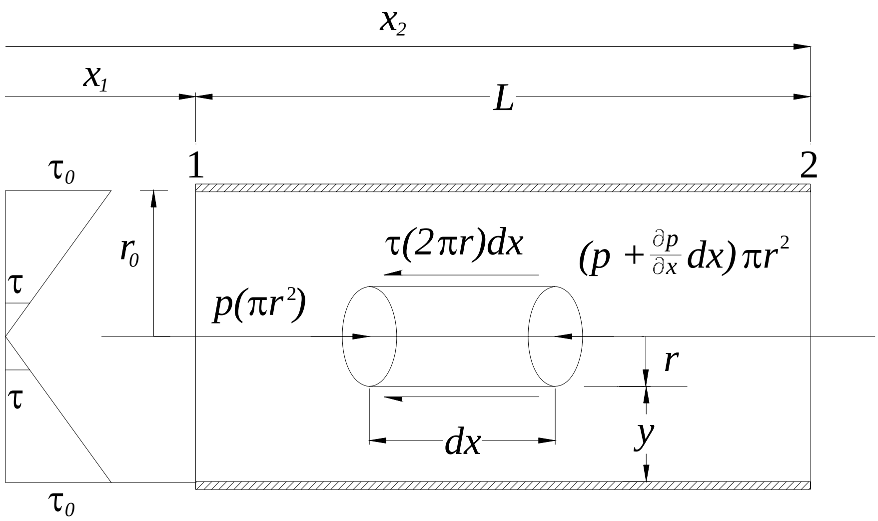


Fig. 7.5

A concentric cylinder of fluid is chosen as a free body (Fig. 7.5). Since the laminar motion of fluid is steady, the momentum equation for the flow of fluid through the chosen free body (in the absence of gravitational forces) is,

$$p\pi r^2 - \left(p + \frac{\partial p}{\partial x}dx\right)\pi r^2 - \tau(2\pi r)dx = 0$$

Which after simplifying, becomes,

$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \tag{7.11}$$

The pressure gradient $\partial p/\partial x$ in the direction of flow depends on x only for any given case of laminar flow. The minus sign indicates a decrease of fluid pressure in the direction of flow in a horizontal pipe, since flow work must be performed on the free body to compensate for the frictional resistance to the flow. Equ. (7.11) shows that the shearing stress is zero at the center of pipe (r=0) and increase linearly with the distance r from the center, attaining its maximum value, $\tau_0=(-\partial p/\partial x)(r_0/2)$, at the pipe wall ($r=r_0$).

In accordance with the first assumption, τ equals $\mu(\partial u/\partial y)$. Since y equals r_0-r, it follows ∂y equals $-\partial r$. Newton's law of viscosity then becomes

$$\tau = -\mu\frac{\partial u}{\partial r} \tag{7.12}$$

in which the minus sign predicts mathematically that u decreases with r. By combining Eqs. (7.11) and (7.12),

$$-\mu\frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x}\frac{r}{2}$$

or

$$\partial u = \frac{1}{2\mu}\frac{\partial p}{\partial x}rdr \tag{7.13}$$

Since $\partial p/\partial x$ is not a function of r, the integration of this differential equation with respect to r then yields the expression for point velocity u:

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}r^2 + C$$

The integration constant C can be evaluated by means of the second assumption, that is, u=0 at $r=r_0$= Therefore, $C = -(\partial p/\partial x)r_0^2/4\mu$, and

$$u = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left(r_0^2 - r^2\right) \tag{7.14}$$

Which is an equation of parabola.

The point velocity varies parabolically along a diameter, and the velocity distribution is a paraboloid of revolution for laminar flow in a straight circular pipe (Fig. 7.6).

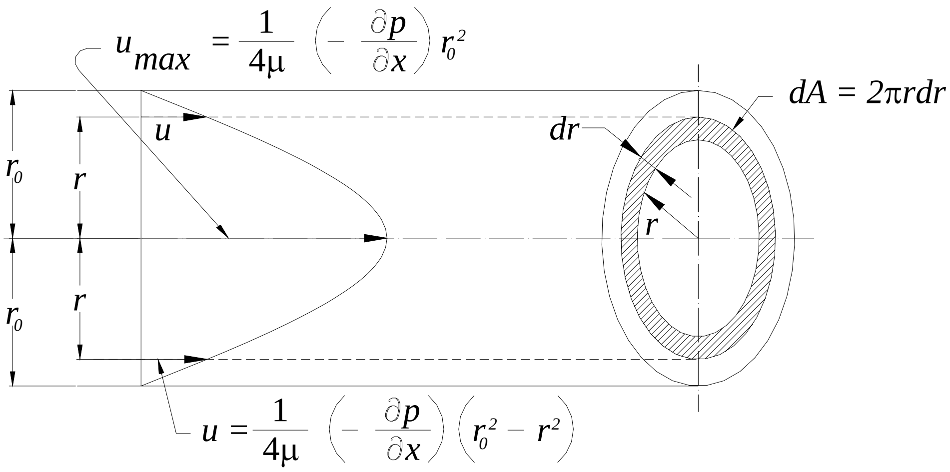


Fig. 7.6

The maximum point velocity u_{max} occurs at the center of the pipe and has the magnitude of

$$u_{\max} = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)r_0^2 \tag{7.15}$$

Thus, from Eqs. (7.14) and (7.15), the point velocity can also be expressed of the maximum point velocity as

$$u = u_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right] \tag{7.16}$$

The volumetric rate of flow Q through any cross section of radius r_0 is obtained by the integration of

$$dq = udA = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left(r_0^2 - r^2\right)(2\pi r)dr$$

as shown in Fig. 7.6. Hence,

$$\begin{aligned} Q &= \frac{\pi}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\int_0^{r_0}\left(r_0^2 - r^2\right)dr \\ &= \frac{\pi}{8\mu}\left(-\frac{\partial p}{\partial x}\right)r_0^4 \end{aligned} \tag{7.17}$$

and the average velocity V is Q/A, or

$$V = \frac{Q}{\pi r_0^2} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)r_0^2 \tag{7.18}$$

comparison of Eqs. (7.15) and (7.18) reveals that

$$V = \frac{u_{\max}}{2} \qquad (7.19)$$

Equ. (7.18) may be arranged in the following form

$$-\partial p = \frac{8\mu V}{r_0^2}\partial x$$

and then integrated with respect to x for any straight stretch of pipe between x_1 and x_2; $L=x_2-x_1$ (Fig. 7.5). Hence,

$$-\int_{p_1}^{p_2}\partial p = \frac{8\mu V}{r_0^2}\int_{x_1}^{x_2}\partial x$$

and, since D equals $2r_0$,

$$p_1 - p_2 = \frac{8\mu VL}{r_0^2} = \frac{32\mu VL}{D^2} \qquad (7.20)$$

This is usually referred to as the Hagen-Poiseuille equation.

EXAMPLE 7.1: A straight stretch of horizontal pipe having a diameter of 5 cm is used in the laboratory to measure the viscosity of crude oil (γ = 0.93 t/m^3). During a test run a pressure difference of 1.75 t/m^2 is obtained from two pressure gages, which are located 6 m apart on the pipe. Oil is allowed to discharge into a weighing tank, and a total of 550 kg of oil is collected for a duration of 3 min. Determine the viscosity of the oil.

SOLUTION: The discharge of oil flow in the pipe is

$$Q = \frac{0.550}{0.93\times 3\times 60} = 0.0033\, m^3/\sec$$

The average velocity is then

$$V = \frac{Q}{A} = \frac{4\times 0.0033}{\pi\times 0.05^2} = 1.69\, m/\sec$$

From Equ. (7.20)

$$\mu = \frac{(p_1 - p_2)D^2}{32VL}$$

$$\mu = \frac{1.75\times 0.05^2}{32\times 1.69\times 6} = 1.35\times 10^{-5}\, t\sec/m^2$$

CHAPTER 8

DIMENSIONAL ANALYSIS

8.1 INTRODUCTION

Dimensional analysis is one of the most important mathematical tools in the study of fluid mechanics. It is a mathematical technique, which makes use of the study of dimensions as an aid to the solution of many engineering problems. The main advantage of a dimensional analysis of a problem is that it reduces the number of variables in the problem by combining dimensional variables to form non-dimensional parameters.

By far the simplest and most desirable method in the analysis of any fluid problem is that of direct mathematical solution. But, most problems in fluid mechanics such complex phenomena that direct mathematical solution is limited to a few special cases. Especially for turbulent flow, there are so many variables involved in the differential equation of fluid motion that a direct mathematical solution is simply out of question. In these problems dimensional analysis can be used in obtaining a functional relationship among the various variables involved in terms of non-dimensional parameters.

Dimensional analysis has been found useful in both analytical and experimental work in the study of fluid mechanics. Some of the uses are listed:

1) Checking the dimensional homogeneity of any equation of fluid motion.
2) Deriving fluid mechanics equations expressed in terms of non-dimensional parameters to show the relative significance of each parameter.
3) Planning tests and presenting experimental results in a systematic manner.
4) Analyzing complex flow phenomena by use of scale models (model similitude).

8.2 DIMENSIONS AND DIMENSIONAL HOMOGENEITY

Scientific reasoning in fluid mechanics is based quantitatively on concepts of such physical phenomena as length, time, velocity, acceleration, force, mass, momentum, energy, viscosity, and many other arbitrarily chosen entities, to each of which a unit of measurement has been assigned. For the purpose of obtaining a numerical solution, we adopt for computation the quantities in SI or MKS units. In a more general sense, however, it is desirable to adopt a consistent dimensional system composed of the smallest number of dimensions in terms of which all the physical entities may be expressed. The fundamental dimensions of mechanics are length [L], time [T], mass [M], and force [F], related by Newton's second law of motion, F = ma.

Dimensionally, the law may also be written as,

$$[F] = \left[\frac{ML}{T^2}\right] \qquad \text{or} \qquad \left[\frac{FT^2}{ML}\right] = 1 \tag{8.1}$$

Which indicates that when three of the dimensions are known, the fourth may be expressed in the terms of the other three. Hence three independent dimensions are sufficient for any physical phenomenon encountered in Newtonian mechanics. They are usually chosen as either [MLT] (mass, length, time) or [FLT] (force, length, time). For example, the specific mass (ρ) may be expressed either as $[M/L^3]$or as $[FT^2/L^4]$, and a fluid pressure (p), which is commonly expressed as force per unit area $[F/L^2]$ may also be expressed as $[ML/T^2]$ using the (mass, length, time) system. A summary of some of the entities frequently used in fluid mechanics together with their dimensions in both systems is given in Table 8.1.

TABLE 8.1
ENTITIES COMMONLY USED IN FLUID MECHANICS
AND THEIR DIMENSIONS

Entity	**MLT System**	**FLT System**
Length (L)	L	L
Area (A)	L^2	L^2
Volume (V)	L^3	L^3
Time (t)	T	T
Velocity (v)	LT^{-1}	LT^{-1}
Acceleration (a)	LT^{-2}	LT^{-2}
Force (F) and weight (W)	MLT^{-2}	F
Specific weight (γ)	$ML^{-2}T^{-2}$	FL^{-3}
Mass (m)	M	$FL^{-1}T^{-2}$
Specific mass (ρ)	ML^{-3}	$FL^{-4}T^2$
Pressure (p) and stress (τ)	$ML^{-1}T^{-2}$	FL^{-2}
Energy (E) and work	ML^2T^{-2}	FL
Momentum (mv)	MLT^{-1}	FT
Power (P)	ML^2T^{-3}	FLT^{-1}
Dynamic viscosity (μ)	$ML^{-1}T^{-1}$	$FL^{-2}T$
Kinematic viscosity (υ)	L^2T^{-1}	L^2T^{-1}

With the selection of three independent dimensions –either [MLT] or [FLT]- it is possible to express all physical entities of fluid mechanics. An equation which expresses the physical phenomena of fluid motion must be both algebraically correct and dimensionally homogenous. A dimensionally homogenous equation has the unique characteristic of being independent of units chosen for measurement.

Equ. (8.1) demonstrates that a dimensionally homogenous equation may be transformed to a non-dimensional form because of the mutual dependence of fundamental dimensions. Although it is always possible to reduce dimensionally homogenous equation to a non-dimensional form, the main difficulty in a complicated flow problem is in setting up the correct equation of motion. Therefore, a special mathematical method called *dimensional analysis* is required to determine the functional relationship among all the variables involved in any complex phenomenon, in terms of non-dimensional parameters.

8.3 DIMENSIONAL ANALYSIS

The fact that a complete physical equation must be dimensionally homogenous and is, therefore, reducible to a functional equation among non-dimensional parameters forms the basis for the theory of dimensional analysis.

8.3.1 Statement of Assumptions

The procedure of dimensional analysis makes use of the following assumptions:

1) It is possible to select m independent fundamental units (in mechanics, m=3, i.e., length, time, mass or force).
2) There exist n quantities involved in a phenomenon whose dimensional formulae may be expressed in terms of m fundamental units.
3) The dimensional quantity A_0 can be related to the independent dimensional quantities A_1, A_2, ..., A_{n-1} by,

$$A_0 = F\left(A_1, A_2, ..., A_{n-1}\right) = KA_1^{y_1} A_2^{y_2} ......A_{n-1}^{y_{n-1}} \tag{8.2}$$

Where K is a non-dimensional constant, and y_1, y_2,.....,y_{n-1} are integer components.

4) Equ. (8.2) is independent of the type of units chosen and is dimensionally homogenous, i.e., the quantities occurring on both sides of the equation must have the same dimension.

EXAMPLE 8.1: Consider the problem of a freely falling body near the surface of the earth. If x, w, g, and t represent the distance measured from the initial height, the weight of the body, the gravitational acceleration, and time, respectively, find a relation for x as a function of w, g, and t.

SOLUTION: Using the fundamental units of force F, length L, and time T, we note that the four physical quantities, A_0=x, A_1=w, A_2=g, and A_3=t, involve three fundamental units; hence, m=3 and n=4 in assumptions (1) and (2). By assumption (3) we assume a relation of the form:

$$x = F\left(w, g, t\right) = Kw^{y_1} g^{y_2} t^{y_3} \qquad \text{(a)}$$

Where K is an arbitrary non-dimensional constant.

Let [·] denote “dimensions of a quantity”. Then the relation above can be written (using assumption (4)) as,

$$\left[x\right] = \left[w\right]^{y_1} \left[g\right]^{y_2} \left[t\right]^{y_3}$$

or

$$F^0 L^1 T^0 = \left(F\right)^{y_1} \left(LT^{-2}\right)^{y_2} \left(T\right)^{y_3} = F^{y_1} L^{y_2} T^{-2y_2 + y_3}$$

Equating like exponents, we obtain

$$F : 0 = y_1$$
$$L : 1 = y_2$$
$$T : 0 = -2y_2 + y_3 \quad \text{or} \quad y_3 = 2y_2 = 2$$

Therefore, Equ. (a) becomes

$$x = Kw^0 g^1 t^2$$

or

$$x = Kgt^2$$

According to the elementary mechanics we have $x=gt^2/2$. The constant K in this case is ½, which cannot be obtained from dimensional analysis.

EXAMPLE 8.2: Consider the problem of computing the drag force on a body moving through a fluid. Let D, ρ, μ, l, and V be drag force, specific mass of the fluid, dynamic viscosity of the fluid, body reference length, and body velocity, respectively.

SOLUTION: For this problem m=3, n=5, A_0=D, A_1=ρ, A_2=μ, A_3=l, and A_4=V. Thus, according to Equ (8.2), we have

$$D = F(\rho, \mu, l, V) = K\rho^{y_1}\mu^{y_2}l^{y_3}V^{y_4} \qquad \text{(a)}$$

or

$$[D] = [\rho]^{y_1}[\mu]^{y_2}[l]^{y_3}[V]^{y_4}$$
$$F^1L^0T^0 = \left(FL^{-4}T^2\right)^{y_1}\left(FL^{-2}T\right)^{y_2}(L)^{y_3}\left(LT^{-1}\right)^{y_4}$$
$$F^1L^0T^0 = F^{y_1+y_2}L^{-4y_1-2y_2+y_3+y_4}T^{2y_1+y_2-y_4}$$

Equating like exponents, we obtain

$$F : 1 = y_1 + y_2$$
$$L : 0 = -4y_1 - 2y_2 + y_3 + y_4$$
$$T : 0 = 2y_1 + y_2 - y_4$$

In this case we have three equations and four unknowns. Hence, we can only solve for three of the unknowns in terms of the fourth unknown (a one-parameter family of solutions exists). For example, solving for y_1, y_3 and y_4 in terms of y_2, one obtains

$$y_1 = 1 - y_2$$
$$y_3 = 2 - y_2$$
$$y_4 = 2 - y_2$$

The required solution is

$$D = K\rho^{1-y_2}\mu^{y_2} l^{2-y_2} V^{2-y_2}$$

or

$$D = (2K)\left(\frac{\rho V l}{\mu}\right)^{-y_2}\left(\frac{\rho V^2}{2}\right) l^2$$

If the Reynolds number is denoted by Re=ρVl/μ, dynamic pressure by q=$\rho V^2/2$, and area by A=l^2, we have

$$D = \frac{2K}{(\mathrm{Re})^{y_2}} qA = C_D qA$$

where

$$C_D = \frac{2K}{(\mathrm{Re})^{y_2}}$$

Theoretical considerations show that for laminar flow

$$2K = 1.328 \qquad \text{and} \qquad y_2 = \frac{1}{2}$$

8.3.2 Buckingham-π (Pi) Theorem

It is seen from the preceding examples that m fundamental units and n physical quantities lead to a system of m linear algebraic equations with n unknowns of the form

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + ...+ a_{1n}y_n &= b_1 \\ a_{21}y_1 + a_{22}y_2 + ...+ a_{2n}y_n &= b_2 \\ .... & \\ a_{m1}y_1 + a_{m2}y_2 + ...+ a_{mn}y_n &= b_m \end{aligned} \qquad (8.3)$$

or, in matrix form,

$$Ay = b \qquad (8.4)$$

where

$$A = \begin{bmatrix} a_{11}a_{12}.......a_{1n} \\ a_{12}a_{22}.......a_{2n} \\ .... \\ a_{m1}a_{m2}......a_{mn} \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ y_n \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix}$$

A is referred to as the coefficient matrix of order m×n, and y and b are of order n×1 and m×1 respectively.

The matrix A in Equ. (8.4) is rectangular and the largest determinant that can be formed will have the order n or m, whichever is smaller. If any matrix C has at least one determinant of order r, which is different from zero, and nonzero determinant of order greater than r, then the matrix is said to be of rank r, i.e.,

$$R(C) = r \qquad (8.5)$$

In order to determine the condition for the solution of the linear system of Equ. (8.3) it is convenient to define the rank of the augmented matrix B. The matrix B is defined as

$$B = A^b = \begin{bmatrix} a_{11}a_{12}.......a_{1n}b_1 \\ a_{21}a_{22}.......a_{2n}b_2 \\ .... \\ a_{m1}a_{m2}.......a_{mn}b_m \end{bmatrix} \qquad (8.6)$$

For the solution of the linear system in Equ. (8.3), three possible cases arise:

1) R (A)<R (B): No solution exists,
2) R(A)=R(B)= r = n: A unique solution exits,
3) R (A)=R (B)=r<n: An infinite number of solutions with (n-r) arbitrary unknowns exist.

Example 8.2 falls in case (3) where

$$R(A) = R(B) = 3 < n = 4 \quad \text{and} \quad (n-r) = (4-3) = 1$$

an arbitrary unknown exits.

The mathematical reasoning above leads to the following Pi theorem due to Buckingham.

Let n quantities A_1, A_2,…..,A_n be involved in a phenomenon, and their dimensional formulae be described by (m<n) fundamental units. Let the rank of the augmented matrix B be R (B)= r ≤ m. Then the relation

$$F_1(A_1, A_2, ..., A_n) = 0 \qquad (8.7)$$

is equivalent to the relation

$$F_2(\pi_1, \pi_2, ..., \pi_{n-r}) = 0 \qquad (8.8)$$

Where π_1, π_2,…..,π_{n-r} are dimensionless power products of A_1, A_2,…..,A_n taken r+1 at a time.

Thus, the Pi theorem allows one to take n quantities and find the minimum number of non-dimensionless parameter, π_1, π_2,….., π_{n-r} associated with these n quantities.

8.3.2.1 Determination of Minimum Number of π Terms

In order to apply the Buckingham π Theorem to a given physical problem the following procedure should be used:

Step 1. Given n quantities involving m fundamental units, set up the augmented matrix B by constructing a table with the quantities on the horizontal axis and the fundamental units on the vertical axis. Under each quantity list a column of numbers, which represent the powers of each fundamental units that makes up its dimensions. For example,

	ρ	p	d	Q
F	1	1	0	1
L	-4	-2	1	3
T	2	0	0	-1

Where ρ, p, d, and Q are the specific mass, pressure, diameter, and discharge, respectively. The resulting array of numbers represents the augmented matrix B in Buckingham's π Theorem, i.e.,

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -4 & -2 & 1 & 3 \\ 2 & 0 & 0 & -1 \end{bmatrix}$$

The matrix B is sometimes referred to as dimensional matrix.

Step 2. Having constructed matrix B, find its rank. From step1, since

$$\begin{vmatrix} 1 & 1 & 0 \\ -4 & -2 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 1 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step3. Having determined the number of π dimensionless (n-r) terms, following rules are used to combine the variables to form π terms.

a) From the independent variables select certain variables to use as repeating variables, which will appear in more than π term. The repeating variables should contain all the dimensions used in the problem and be quantities, which are likely to have substantial effect on the dependent variable.
b) Combine the repeating variables with remaining variables to form the required number of independent dimensionless π terms.
c) The dependent variable should appear in one group only.
d) A variable that is expected to have a minor influence should appear in one group only.

Define π_1 as a power product of r of the n quantities raised to arbitrary integer exponents and any one of the excluded (n-r) quantities, i.e.,

$$\pi_1 = A_1^{y_{11}} A_2^{y_{12}} .....A_r^{y_{1r}} A_{r+1}$$

Step 4. Define π_2, π_3,....., π_{n-r} as power products of the same r quantities used in step 3 raised to arbitrary integer exponents but a different excluded quantity for each π term, i.e.,

$$\pi_2 = A_1^{y_{21}} A_2^{y_{22}} ....A_r^{y_{2r}} A_{r+2}$$
$$\pi_3 = A_1^{y_{31}} A_2^{y_{32}} ....A_r^{y_{3r}} A_{r+3}$$
$$...$$
$$\pi_{n-r} = A_1^{y_{n-r,1}} A_2^{y_{n-r,2}} .....A_r^{y_{n-r,r}} A_n$$

Step 5. Carry out dimensional analysis on each π term to evaluate the exponents.

EXAMPLE 8.3: Rework Example 8.1 using the π theorem.

SOLUTION:

Step 1. With F, L, and T as the fundamental units, the dimensional matrix of the quantities w, g, t and x is,

	W	g	t	x
F	1	0	0	0
L	0	1	0	1
T	0	-2	1	0

Where

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

Step 2. Since

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step 3. Arbitrarily select x, w, and g as the r = m= 3 base quantities. The number n-r of independent dimensional products that can be formed by the four quantities is therefore 1, i.e.,

$$\pi_1 = x^{y_{11}} w^{y_{12}} g^{y_{13}} t$$

Step 4. Dimensional analysis gives,

$$[\pi_1] = [x]^{y_{11}} [w]^{y_{12}} [g]^{y_{13}} [t]$$

or

$$F^0 L^0 T^0 = (L)^{y_{11}} (F)^{y_{12}} (LT^{-2})^{y_{13}} (T)$$

Which results in

$$y_{11} = -\frac{1}{2},\ y_{12} = 0, \text{ and } y_{13} = \frac{1}{2}$$

Hence,

$$\pi_1 = \sqrt{\frac{gt^2}{x}}$$

EXAMPLE 8.4: Rework Example 8.2 using the π theorem.

SOLUTION:

Step 1. With F, L and T as the fundamental units, the dimensional matrix of the quantities D, ρ, μ, l, and V is.

	D	ρ	μ	l	V
F	1	1	1	0	0
L	0	-4	-2	1	1
T	0	2	1	0	-1

Where

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 1 \\ 0 & 2 & 1 & 0 & -1 \end{bmatrix}$$

Step2. Since

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 1 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step 3. Select l, V, and ρ as the r=3 base quantities. By the π-Theorem (n-r),(5-3)=2 π terms exist.

$$\pi_1 = l^{y_{11}} V^{y_{12}} \rho^{y_{13}} D$$
$$\pi_2 = l^{y_{21}} V^{y_{22}} \rho^{y_{23}} \mu$$

Step 4. Dimensional analysis gives

y_{11}=-2, y_{12}=-2, y_{13}= -1

y_{21}=-1, y_{22}=-1, y_{23}=-1

Hence,

$$\pi_1 = \frac{D}{\rho V^2 l^2}$$
$$\pi_2 = \frac{\mu}{\rho V l}$$

8.4 THE USE OF DIMENSIONLESS π-TERMS IN EXPERIMENTAL INVESTIGATIONS

Dimensional analysis can be of assistance in experimental investigation by reducing the number of variables in the problem. The result of the analysis is to replace an unknown relation between n variables by a relationship between a smaller numbers, n-r, of dimensionless π-terms. Any reduction in the number of variables greatly reduces the labor of experimental investigation. For instance, a function of one variable can be plotted as a single curve constructed from a relatively small number of experimental observations, or the results can be represented as a single table, which might require just one page.

A function of two variables will require a chart consisting of a family of curves, one for each value of the second variable, or, alternatively the information can be presented in the form of a book of tables. A function of three variables will require a set of charts or a shelf-full of books of tables.

As the number of variables increases, the number of observations to be taken grows so rapidly that the situation soon becomes impossible. Any reduction in the number of variables is extremely important.

Considering, as an example, the resistance to flow through pipes, the shear stress or resistance R per unit area at the pipe wall when fluid of specific mass ρ and dynamic viscosity μ flows in a smooth pipe can be assumed to depend on the velocity of flow V and the pipe diameter D. Selecting a number of different fluids, we could obtain a set of curves relating frictional resistance (measured as $R/\rho V^2$) to velocity, as shown in Fig. 8.1.

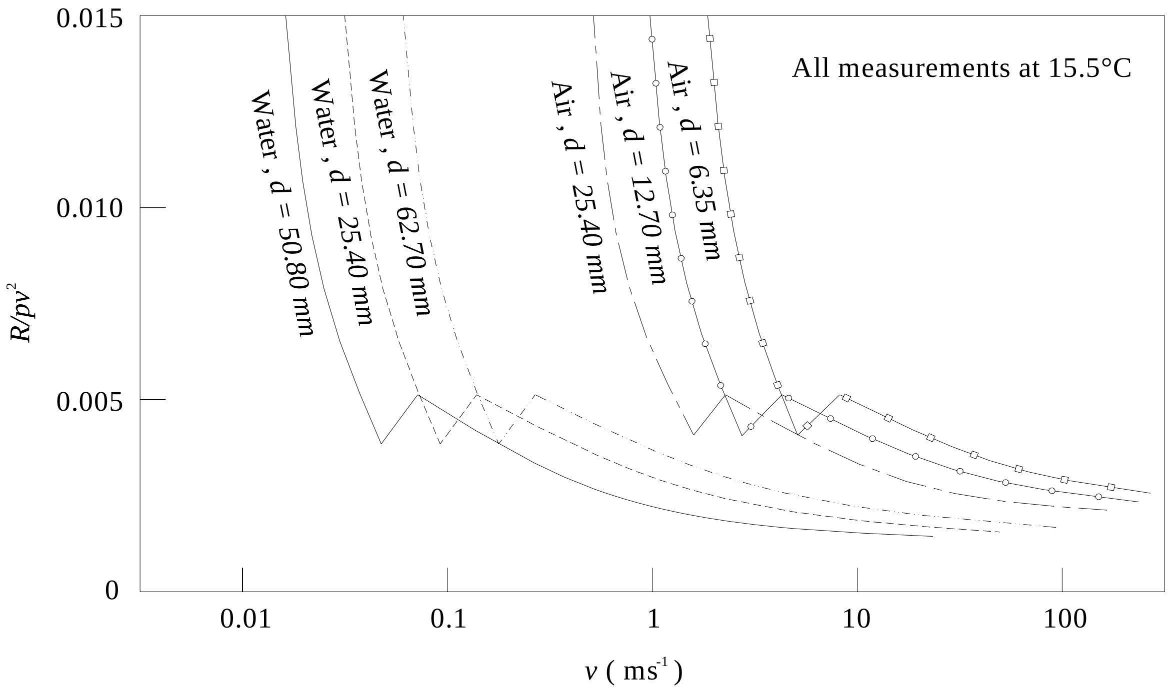


Fig. 8.1

Such a set of curves would be of limited value both for use and for obtaining a proper understanding of the problem. However, it can be shown by dimensional analysis that the relationship can be reduced to the form

$$\frac{R}{\rho V^2} = \phi\left(\frac{\rho VD}{\mu}\right) = \phi(\mathrm{Re})$$

or, using the Darcy resistance coefficient $f = 2R/\rho V^2$,

$$f = \phi\left(\frac{\rho VD}{\mu}\right) = \phi(\mathrm{Re})$$

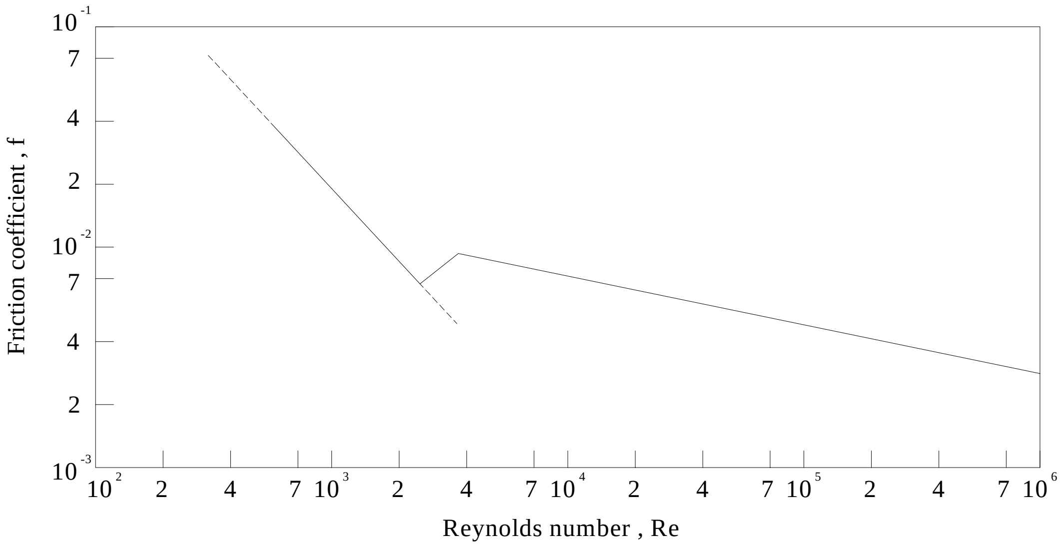


Fig. 8.2

If the experimental points in Fig. 8.1 are used to construct a new graph of Log (f) against Log (Re) the separate sets of experimental data combine to give a single curve as shown in Fig. 8.2. For low values of Reynolds number, when flow is laminar, the slope of this graph is (-1) and $f = 16/Re$, while for turbulent flow at higher values of Reynolds number, $f = 0.08(Re)^{-1/4}$.